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Mary Melissa Rodd BA MA

MATHEMATICAL WARRANTS, OBJECTS AND ACTIONS
IN HIGHER SCHOOL MATHEMATICS

Thesis submitted for the degree of Doctor of Philosophy

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ABSTRACT

'Higher school mathematics' connotes typical upper secondary school and early college mathematics. The mathematics at this level is characterised by moves to ① rigour in justification, ② abstraction in content and ③ fluency in symbolic manipulation.

This thesis investigates these three transitions - towards rigour, abstraction, and fluency-using philosophical method: for each of the three transitions a proposition is presented and arguments are given in favour of that proposition. These arguments employ concepts and results from contemporary English language-medium philosophy and also rely crucially on classroom issues or accounts of mathematical experience both to elucidate meaning and for the domain of application. These three propositions, with their arguments, are the three sub-theses at the centre of the thesis as a whole.

The first of these sub-theses ① argues that logical deduction, quasi-empiricism and visualisation are mathematical warrants, while authoritatively based justification is essentially non-mathematical. The second sub-thesis ② argues that the reality of mathematical entities of the sort encountered in the higher school mathematics curriculum is actual not metaphoric. The third sub-thesis ③ claims that certain 'mathematical action' can be construed as non-propositional mathematical knowledge. The application of these general propositions to mathematics in education yields the following: 'coming to know mathematics' involves: ① using mathematical warrants for justification and self-conviction; ② ontological commitment to mathematical objects; and ③ developing a capability to execute some mathematical procedures automatically.

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1. CHAPTER : INTRODUCTION.....	7
1.1 THE STARTING POINT	7
1.2 STATEMENT OF THE THESIS	7
1.3 BRIEF RATIONALE FOR THESIS STRUCTURE	8
1.4 PHILOSOPHY AND EXPERIENCE AS INTEGRAL TO INVESTIGATING THE BASIC QUESTION.....	8
1.5 DIALOGUES.....	10
1.5.1 <i>On belief</i>	10
1.5.2 <i>On mathematical objects</i>	11
1.5.3 <i>On notation</i>	12
1.5.4 <i>On proof</i>	14
1.5.5 <i>Wis en zeker?</i>	15
1.5.6 <i>On the role of these dialogues</i>	17
2. CHAPTER 2: METHODOLOGY	18
2.1 INTRODUCTION	18
2.2 A STUDENT'S PROGRESS?.....	19
2.2.1 <i>Initial experience of research</i>	19
2.2.2 <i>The next stage</i>	21
2.2.3 <i>A change of method after reflection on earlier work</i>	23
2.3 EDUCATIONAL RESEARCH AND PHILOSOPHY.....	24
2.3.1 <i>On social science and educational research</i>	25
2.3.2 <i>Some meanings of 'philosophy'</i>	26
2.3.3 <i>From first exposure to current use of 'philosophy'</i>	29
2.4 PRACTITIONER RESEARCH AND PHILOSOPHY.....	30
2.4.1 <i>Practitioner research</i>	30
2.4.2 <i>Philosophical research</i>	32
2.4.3 <i>Considerations about 'validity'</i>	35
2.5 SUMMARY	36
3. CHAPTER 3: PHENOMENA.....	38
3.1 INTRODUCTION	38
3.2 MATHEMATICAL EXPERIENCE: PERSONAL ACCOUNTS.....	39
3.2.1 <i>On angle trisection</i>	40
3.2.1.1 <i>Mathematical evidence for non-trisection</i>	40
3.2.1.2 <i>Meta-mathematical reflections</i>	41
3.2.1.3 <i>Trisection: how close can you get?</i>	43
3.2.1.4 <i>For further philosophical investigation:</i>	43
3.2.2 <i>The group exists: understanding isomorphism</i>	44
3.2.2.1 <i>Mathematical evidence for abstract group existence</i>	45
3.2.2.2 <i>Meta-mathematical reflections</i>	46

3.2.2.3 How to represent the structure so that the abstract form is recognisable?	48
3.2.2.4 For further philosophical investigation	48
3.2.3 <i>That looks like an ellipse</i>	48
3.2.3.1 Mathematical evidence for the locus being an ellipse	49
3.2.3.2 Meta-mathematical Reflections	51
3.2.3.3 Algebra and geometry	53
3.2.3.4 For further philosophical investigation	53
3.2.4 <i>The Golden Circle property</i>	53
3.2.4.1 Mathematical evidence for the 'golden circle property'	54
3.2.4.2 Meta-mathematical reflections	55
3.2.4.3 A task: why is it a 'Golden egg'?	56
3.2.4.4 For further philosophical investigation	57
3.3 MATHEMATICAL CONTENT: EXAMPLES AT THE 'HIGHER SCHOOL LEVEL'	57
3.3.1 <i>Axiomatisation</i>	58
3.3.1.1 topological equivalence	58
3.3.1.2 axiomatization: rule recognising and following	59
3.3.2 <i>Modelling</i>	60
3.3.3 <i>Infinite processes, infinite objects</i>	61
3.3.4 <i>Symbolic manipulation</i>	62
3.4 PEDAGOGICAL MATHEMATICAL KNOWLEDGE EXEMPLIFIED AND ANALYSED	64
3.4.1 <i>Forward planning for differential equations</i>	64
3.4.2 <i>Responding in the moment to students' mathematical work in this lesson</i>	66
3.5 SUMMARY	67
4. CHAPTER 4: PHILOSOPHICAL PRELIMINARIES	69
4.1 INTRODUCTION	69
4.2 STATEMENT OF OVERALL THESIS	70
4.2.1 <i>Some underlying issues</i>	71
4.3 PRELIMINARIES FOR I	71
4.3.1 <i>Theories of truth relevant to mathematics in education</i>	72
4.3.1.1 Epistemic or psychological	73
4.3.2 <i>Theories of truth</i>	75
4.3.2.1 Referential realism	76
4.3.2.2 Epistemic theories of truth	77
4.3.2.3 Correspondence theory	78
4.3.2.4 Realist theories of truth	80
4.3.3 <i>Summary</i>	81
4.4 PRELIMINARIES FOR II	82
4.4.1 <i>Realism and anti-realism</i>	83
4.4.1.1 Realism	85
4.4.1.2 Conceptualism: Dummett's semantic logicism	87

4.4.1.3 Nominalism 90

4.5 PRELIMINARIES FOR III 92

4.5.1 *On Intentionality*..... 93

4.5.2 *Intention in education* 95

4.6 SUMMARY 95

5. CHAPTER 5: MATHEMATICAL WARRANTS..... 97

5.1 INTRODUCTION 97

5.1.1 *Justification is part of mathematics* 97

5.1.2 *Outline of the chapter* 98

5.1.3 *Exemplification of thesis in educational contexts* 98

5.2 KNOWLEDGE AND BELIEF. 99

5.2.1 *What is a ‘warrant’?*..... 101

5.3 EPISTEMOLOGICAL THEORY 103

5.3.1 *Goldman* 105

5.3.1.1 Reliability of belief causes and belief justifications: Goldman’s theory 105

5.3.1.2 Initial beliefs have causes 106

5.3.1.3 Beliefs to knowledge: discrimination and reliability 107

5.3.1.4 Beliefs to knowledge: justification 108

5.3.2 *Gettier problems.* 112

5.3.3 *Moser’s theory* 113

5.4 WHAT ARE MATHEMATICAL WARRANTS? 116

5.4.1 *An example of warrant and non-warrant*..... 117

5.4.1.1 A task with which to explore warrants: 120

5.4.2 *Hanna* 120

5.4.2.1 Rigorous proof, per se, is no warrant..... 120

5.4.3 *Simple logical inference as a basic mathematical warrant* 122

5.4.4 *Quasi-empiricism*..... 126

5.4.4.1 What is quasi-empiricism? 128

5.4.4.2 Is mathematics quasi-empirical?..... 130

5.4.4.3 ‘Naïve abstractionism’ and quasi-empiricism..... 132

5.4.4.4 Quasi-empiricism as dialectical 133

5.4.4.5 Quasi-empiricism and the classroom..... 134

5.4.5 *Can visualisation be a mathematical warrant?* 137

5.4.5.1 What is Giaquinto’s visualizing? 137

5.4.5.2 Anecdotal evidence of some of these requirements 139

5.4.5.3 Visualization includes mental-manipulation 140

5.4.5.4 Visualizing and time..... 141

5.4.5.5 ‘Inner experiments’ are not visualizations 143

5.4.6 *A visualisation (for the experience and the post-experience reflection)* 144

5.5 BELIEF AND ITS GENESIS IN LEARNING MATHEMATICS 145

5.5.1 Towards analysing student belief.....	145
5.5.2 Some types of student belief.....	146
5.5.2.1 Belief development.....	148
5.5.3 Data from students' work: some student justifications of the angle in a semi-circle property	149
5.5.4 Belief and proof: Coe and Ruthven's study on students' proof practices	152
5.6 CONCLUSION	153
6. CHAPTER 6: MATHEMATICAL OBJECTS.....	155
6.1 INTRODUCTION	155
6.1.1 Outline of the chapter	156
6.2 TOWARDS AN ARGUMENT FOR REALISM IN DISCUSSING MATHEMATICS LEARNING	157
6.2.1 Rejection of nominalism.....	157
6.2.1.1 Mathematical generalities; scientific particularities.....	158
6.2.1.2 Just notations: 'black boxes' and meaning	159
6.2.2 Rejection of conceptualism	159
6.2.2.1 Outline of Ernest's conception of 'mathematical objects'	161
6.2.2.2 An argument against this conception of mathematical objects	163
6.2.3 Towards realism: summary.....	165
6.3 FROM PLATO'S BEGINNINGS	165
6.3.1 On accounts of mathematicians' practice	166
6.3.2 Is platonism the same as realism?	167
6.3.3 Platonism and realism: summary	167
6.4 MADDY'S THEORY.....	168
6.4.1 Background to Maddy's realism.....	169
6.4.1.1 "Quine/Putnamism".....	169
6.4.1.2 Gödel's platonism.....	170
6.4.1.3 Realism and epistemology: the problem of access and the Benacerraf syllogism.....	170
6.4.1.4 Physicalistic platonism	171
6.4.2 Back to realism à la Maddy: Set theoretic realism	173
6.4.2.1 How to perceive a set.....	173
6.4.2.2 Some objections to set theoretic realism.....	175
6.5 OTHER REALIST PHILOSOPHERS	179
6.5.1 Resnik.....	180
6.5.1.1 Pedagogical implications.....	183
6.5.2 Bigelow	183
6.6 KITCHER.....	184
6.6.1 A priority and experience.....	187
6.6.2 Mathematical objects.....	191
6.6.2.1 Kitcher on Newton and Leibniz's 'perceptual beginnings'	194
6.6.2.2 Kitcher's implicit mathematical objects	195

6.6.3	<i>Platonist and non-platonist mathematical realism</i>	196
6.7	AN HISTORICAL PERSPECTIVE ON MATHEMATICAL ONTOLOGY	197
6.7.1	<i>'Intuitionism' and 'naïve abstractionism'</i>	198
6.7.1.1	Gray's ontological distinction	198
6.7.1.2	Abstract objects and 'levels' in mathematical ontology	200
6.7.2	<i>Gray on mathematicians' epistemological evolution</i>	201
6.7.3	<i>Other historians' of mathematics dichotomies</i>	202
6.7.4	<i>Historical evidence for mathematical objects</i>	203
6.8	MATHEMATICAL REALISM IN THE CLASSROOM?	204
6.8.1	<i>Mathematical ontology for higher school mathematics</i>	206
6.9	ONTOLOGICAL COMMITMENT AND LEARNING MATHEMATICS	210
6.9.1	<i>What is 'ontological commitment'?</i>	211
6.9.1.1	Alston	213
6.9.1.2	Quine	214
6.9.1.3	Devitt	215
6.9.1.4	Harré	216
6.9.2	<i>Ontology and ontological commitment</i>	216
6.9.2.1	Consequences of adopting a realist ontology	217
6.9.2.2	Experiential learning of mathematics and ontology	218
6.9.3	<i>Back to pedagogy: negative numbers</i>	219
6.9.3.1	Negative numbers and set-theoretic realism	219
6.9.3.2	Some pupils work with negative numbers	223
6.10	SUMMARY REMARKS	225
7.	CHAPTER 7: MATHEMATICAL ACTION	226
7.1	INTRODUCTION	226
7.1.1	<i>Examples of automaticity</i>	227
7.1.2	<i>Piaget's legacy</i>	229
7.1.3	<i>This chapter's proposition</i>	230
7.1.4	<i>Outline of the chapter</i>	231
7.2	AUTOMATICITY FROM DIFFERENT PERSPECTIVES	231
7.2.1	Krutetskii	234
7.2.2	Mark Johnson	237
7.2.3	John O'Neill	239
7.2.4	Wang Hao	240
7.2.5	Wittgenstein	241
7.3	A CASE FOR AUTOMATICITY AS KNOWLEDGE	243
7.3.1	<i>Automaticity: an example</i>	245
7.3.1.1	Vedic form as mathematical warrant	248
7.3.2	<i>Automaticity as non-propositional knowledge</i>	248
7.3.3	<i>Objections</i>	251

7.4 CAS, CALCULUS AND AUTOMATICITY	254
7.4.1 <i>'The product rule': performance, and perception of structure</i>	256
7.4.1.1 Problem solving and perception of structure.	257
7.4.1.2 Learning about structure with CAS.	259
7.4.2 <i>CAS and capability</i>	260
7.5 SUMMARY: ONLY IN MATHEMATICS	262
8. CHAPTER 8: CONCLUSION	263
8.1 OVERVIEW.....	263
8.2 ON THE COMMENTS ARISING FROM DOING MATHEMATICS	264
8.3 BACK TO THE THEMES OF THE 'TEACHERS' DIALOGUES'	265
8.4 A SAMPLE OF HIGHER SCHOOL MATHEMATICS ANALYSED IN TERMS OF WARRANTS, OBJECTS AND ACTIONS	266
8.4.1 <i>A level algebra question</i>	267
8.4.2 <i>A level calculus question</i>	270
8.4.3 <i>Problem situation from Mechanics</i>	273
8.4.4 <i>Summary</i>	274
8.5 DIRECTIONS FOR FURTHER WORK	274
8.6 "IMAGE AND REALITY: THE OLDEST DISTINCTION OF ALL."	277
9. APPENDICES.....	280
9.1 FOR CHAPTER 5	280
9.2 FOR CHAPTER 6	280
10. BIBLIOGRAPHY.....	281

1. Chapter : Introduction

1.1 The starting point

My naïve starting point was the question: What is it to know mathematics? This thesis is a partial answer to this question which deliberately focuses on aspects of epistemology and ontology rather than on social or linguistic concerns. The question of the nature of mathematics is an ancient one, different interpretations and responses to the question have been given over the ages. Each approach to the question is inevitably coloured by the historical circumstances and personal interests of the investigator. I shall argue for three related propositions each concerned with the nature of mathematics and which are pertinent to learning mathematics. The approach is philosophical, rather than empirical: I aim to give a general argument and show its educational relevance, rather than analyse some data from an educational context and then generalise.

1.2 Statement of the thesis

The three branches of the thesis are mathematical *warrants*, mathematical *objects* and mathematical *action*. The decision to form my thesis from these came from my analysis of mathematical practice: my own, my students' and published accounts. My interest and expertise is in secondary school and early college mathematics, which I refer to as 'higher school mathematics'.

I shall briefly state the thesis in two ways: Firstly, from an educationalist's perspective and then, in more generality, as in philosophical discourse. Education, by definition and purpose involves change, so, from this perspective, learning mathematics involves changes which can be characterised as follows:

- i. change in distinguishing mathematical warrants: mathematical propositions are warranted in 'mathematical' ways
- ii. change in identifying mathematical objects as existing: ontological commitment is made to the content of mathematical propositions

iii. change in executing mathematical processes - some are automatic: mathematical activity becomes personal action-knowledge.

A different, more general, formulation of this three-branched thesis is based on the epistemological and ontological assertions therein:

- I. the nature of mathematical warrants is logico-deductive, quasi-empirical and, possibly 'visual', but not purely empirical or authoritative
- II. the term 'mathematical object' is not purely metaphorical: philosophical realism allows a sense in which mathematical objects exist
- III. mathematical action, as in the activity of solving an equation, can be an aspect of mathematical knowledge.

1.3 Brief rationale for thesis structure

The structure of the thesis as a whole is not quite standard, for it mixes the structure of a typical mathematics education thesis with that of a philosophy one. After this brief introduction, Chapter 2 sets out the methodology, the rationale of the method used to develop the thesis, the substance of which is that mathematical practice and philosophical reasoning should be the bases of the work. Hence my next chapter presents some mathematical work at the higher school mathematics level, some examples of higher school mathematics and an example of realising pedagogical content knowledge. From reflections on these, the principal philosophical directions for investigation are taken. Chapter 4 is, in effect, an introduction to the philosophical background and Chapters 5, 6 and 7 set out the background literature and the arguments for the propositions on warrant, objects and actions as stated above. Chapter 8 summarises and concludes.

1.4 Philosophy and experience as integral to investigating the basic question

As a mathematics teacher, I used to spend much time working school mathematics problems both publicly, in class, and privately, for preparation. For many years I taught the whole range of comprehensive school pupils including both low attainers and exceptionally high attainers. Throughout this period I was kept occupied, in mind

and body, both with mathematics and the puzzle of how to help others to come to know that which I knew. For every teacher, the job is essentially to assist in that ephemeral transition between not-knowing and knowing. So when a student cries "I've got it!" a philosophically inclined teacher might ask 'What is the nature of this "getting" and this "it"?' The first of these is essentially an epistemological question; it concerns coming to know ("getting"). The second is concerned with ontology; what is "it" in higher school mathematics?

Although I am interested in the academic notions of ontology and epistemology, my mathematics teacher persona constantly reminds me to seek an application of the more abstruse aspects of this enquiry within the practice of teaching mathematics. The desire to entwine theoretical and practical is particularly prevalent in feminist writings that have an intellectual as well as experiential component. For example, from the archeologist-historian Lucy Goodison (1990) we find a clear expression of the intention to link these components:

“Readers may be disconcerted at the way the empirical and the esoteric rub shoulders in these pages....Detailed work does not need to be dry: knowledge can be fed by experience, writing can spring from both thought and passion... I hope that I shall also be able to bridge the gap between insight and enquiry, between head and heart, in the way I write [the book].” (p 4).

While Goodison focuses on the connection between her subject and her passion for that subject, the writer and English professor, bell hooks, emphasises passion in and for teaching:

“those of us who have been intimately engaged as students or teachers with feminist thinking have always recognised the legitimacy of a pedagogy that dares to subvert the mind/body split and allows us to be whole in the classroom, and as a consequence wholehearted” (bell hooks, 1994, p193)

Despite the difference in our subject matters, my aim, like Goodison's and bell hooks', is to bring intellectual enquiry as close as possible to the practice with which I am involved. A purpose of undertaking this research is to strengthen my mathematics teaching. As I have said above, the basic question arose from the practice of teaching

mathematics, is concerned with the experience of doing mathematics and involves reflection on the nature of mathematics and mathematical knowledge.

1.5 Dialogues

In order to orient the reader to the way philosophical ideas may arise within teaching higher school mathematics, I offer the following - light hearted - dialogues involving some puzzling ideas which practising teachers rarely have time to pursue:

Dramatis personae: The mathematics teachers: Miss (M), Sir (S)

All scenes take place in the staff room.

1.5.1 On belief.

Monday 10:48 a.m.

M. I've just started limits with my beginning calculus class, and, as per usual, they tell me that $1/n$ 'never quite' gets to zero and 0.9 'never quite' gets to 1, and so on.

S. Is that because they can't accept that an infinite process yields something nice and familiar like a small finite number?

M. Perhaps. Maybe because we start off experimentally they can't get beyond this 'potentially physically possible phase'.

S. Do you get them to use a spreadsheet? They'd have to be pretty skeptical to disbelieve $1/n$ does get to 0; you can get an awfully long way along the sequence with them!

M. It depends on the computer room availability. They either use that or their calculators to see what the sequence's n th term is for n as large as they want. I also use that 'where will it all end?' page in one of the ATM's A level books¹ as a starter for them exploring limits visually.

S. Is the response to the picture sequence the same as to the number sequence?

M. Pretty much. The stumbling point is that whatever they can draw is inevitably unfinished. And as they say, rightly!, it never could be finished, so they argue the limit is theoretically unobtainable.

S. Most of those pictures in the ATM book you mentioned are representations of geometric progressions. Would it help them to believing limits actually exist if they

¹ Association of Teachers of Mathematics (ATM), (1988, p84)

used the formula for the limit of a GP, having worked out the common ratio and the first term from the drawing?

M. I doubt it! The proof of the formula for the sum to infinity depends upon $(\lim_{n \rightarrow \infty} r^n) = 0$, with $r < 1$, which is one of those things that they tell me never quite gets there!

S. So they believe their senses! Is this incompatible with believing in the existence of limits and other mathematical things that involve infinite processes? Or other abstract mathematical notions?

M. Well it sounds daft to say 'don't believe what you see'. But at some point they need to be able to drop the particulars of the starter situation that they've played with empirically and go with the abstraction rather than the perceptible.

11 a.m., the bell goes.

1.5.2 On mathematical objects.

Tuesday, 3:40 p.m.

M. You look miles away! What about a game of squash? Don't know about you, but I've had quite a day!

S. Sorry, I teach this access evening class Tuesdays. In fact, I was just thinking about what one of the students said to me last session....

M. I wouldn't have thought that class was a major mathematical challenge... What did he or she say?

S. She, Betty. She asked me what minus one *really* was. I didn't know what to say. We'd just spent the entire hour and a half on negative numbers, I thought the session had gone quite well, you know, they got on happily enough, asking a few questions but cranking through the work sheet pretty smoothly. Then Betty says 'but what *is* minus one?' and I felt that she'd pulled the rug from under me.

M. I remember a similar experience from a few years ago when I taught Toby - do you remember him ? - he was always asking what *is* a function? What *is* a set? I was never quite sure whether he was just trying to deflect from getting on with his maths or whether he needed some sort of answer to those questions in order to proceed. He was actually a very good mathematician, but he went off to do philosophy at university.

S. I see what you mean, they are similar questions, although they come from students with very different backgrounds and different expectations about the fruits of their mathematical education. It's like they're asking 'well, okay, I can do this and that, but what is the *stuff* I'm actually working with?'

M. That sort of comes back to what we were talking about yesterday, doesn't it? mathematical stuff can't be touched and seen and heard and so on. Mathematical things are abstract.

- S. What does that mean: 'abstract'? I know we use the term often enough, but now I'm worried that I don't know, in essence, what a mathematical object is.
- M. I'd say that I know what some mathematical objects are, as particular items, but I'd say to try to get a definition of 'mathematical object' in general is a pretty hopeless quest. I mean, nobody knows what all mathematical objects are - how could anyone point to something they had in common that defined them -
- S. unless it was to say, they are what mathematicians use !-
- M. which wouldn't be very helpful for answering questions like Betty's or Toby's.
- S. Coming back to that, have you any ideas about what I should say to Betty tonight?
- M. Well, minus one is the inverse of plus one, additively of course; multiplicatively -
- S. Oh come on, you're just starting to play with a mathematical definition that is starting to sound even more obscure than the original.
- M. But without really knowing that $-1 + 1 = 0$, I don't see how she will be able to understand -1 . And you said that they could do the exercises you set.
- S. So, are you saying that you can't know what a mathematical object is until you can work with it. But surely you can't work with it until you know what it is? I think I'm in a worse muddle than Betty.
- M. Yeah, we could get so confused we'd not get anything done. Ha, P.J. is waving her squash racquet at me...- *fine, P.J.! let's go now* - have fun with Betty et al. this evening, see you tomorrow.

1.5.3 On notation.

Wednesday 12:45 p.m.

- M. These so-called helpful parents! I have been very careful with the notation I've been using in my intermediate calculus class, you know, functional notation whenever possible, acknowledging the alternative Leibnitz notation, but no δx s thank you! Now I've just been with Emma, who is in a dreadful muddle, because her father has gone through derivatives with her with an abundance of δx s and...is there any lunch left?
- S. At this hour! I'll get you a tea, if I can find a cup.
- M. Thanks. I'm wondering if I'm being silly, fussing about the way of writing things, I mean, they're only squiggles on a page, presumably, a derivative is a derivative, no matter how you write the 'd'?
- S. But isn't why we steer away from the δx notation is because that notation tempts us into doing illegitimate things? The usual story that goes with $\frac{\delta y}{\delta x}$ is that it starts off as a ratio of two perfectly decent numbers, then magically changes to a value, $\frac{dy}{dx}$,

at the instant that both of the numbers in the ratio hit zero- would it be obvious that happens together?

M. Sounds like there are different mathematical objects around - the ratio and the derived function...how was your evening class?

S. We got rather side tracked...whether god existed and could we ever know that.. I was well out of my depth. Once they got started there was no stopping them. I was a bit worried as the maths was not getting done. But as they were about to go Betty said how much she'd enjoyed the session and could they take the work I'd planned home. I was relieved! Coming back to the notation, and whether it matters, I'd say that it does matter because different notation lets you do different things.

M. Yes, like those δx s just love to 'go to zero' don't they? I also find that notation acts as a sort of image for me, like, $\frac{dy}{dx}$ is not just a squiggle, it's like a name as well as an instruction, it holds quite a lot of information.

S. Yes, I agree, it is as if these squiggles take on a life of their own: the symbol becomes symbolic - did somebody say that²?

M. Dunno, sounds a bit profound for you. But I find myself encouraging the students to use x for real variables n for integers, as well as functional notation for derivatives, and so on. Isn't it a bit like with young children? my daughter called anything on four legs a 'cat' when she was tiny; she soon made distinctions between pet animals for herself ; some students need more help with picking up the language than others.

S. I don't think mathematical notation is really the same as words in a language because we can get unexpected results from fiddling about with these symbols in a way that can't happen with words.

M. I sort of agree that formalism works, but there is some linguistic part of maths too isn't there?

S. What, more than learning our meaning for 'volume', 'take away', and so on?

M. I think so, like when we see the squiggle $\frac{dy}{dx}$ we can't help but have all sorts of ideas that come into our minds.

S. Like it's 12:59 and I've got *my* intermediate calculus class first thing this afternoon!

1 p.m., the bell goes.

² Mason, 1980, did!

1.5.4 *On proof.*

Thursday 8:17 a.m.

S. I can't find that Cabri disk anywhere! Have you seen it?

M. I thought that the I.T. manager had it, she was going to put Cabri on the network -

S. no can do, it'll cost more. Anyway, thanks, I'll pick it up from her office presently.

M. You could always phone her secretary on his mobile. So, what are you using Cabri for?

S. I'm doing some Euclidean geometry with the class I teach intermediate calculus, some of them will probably continue with maths beyond this institution, so it seems appropriate to do some of this sort of geometry.

M. Helps them with the idea of proof doesn't it?

S. Well, I know what you mean, you can construct neat little proofs of various geometric relationships. But I must say, I was quite surprised at their idea of what a proof was: I gave them this homework the other week - 'show that the angle in a semi-circle is a right angle' and I tried to encourage them to write up more than one way of showing this, indicating which they thought was the most convincing and why.

M. How did it go?

S. Well, I was a bit surprised. All of them who gave in more than one demonstration, included a 'draw 'n measure' approach.

M. Doesn't that just relate to knowledge starting from experience? - though as we were talking about before - its hard to see how to get beyond finite experience.

S. Yes, that's fair enough. But would you have expected them to say that the 'draw 'n measure' was the most convincing?

M. Not really. I'd have thought that they'd all be aware that no diagram can be totally accurate.

S. Oh, they were quite aware of that, but it didn't stop the ones who did a 'draw 'n measure' from thinking it was the most convincing. Anthea, of course, was rather scathing of their measuring business. She'd given me a two line proof - correct, needless to say - and said that she couldn't understand the point of doing more than one 'demonstration'. I told her that Gauss liked to do several proofs of the same thing; that kept her quiet. She'd got three more proofs by the end of the session, although she said that they were all equivalent and Gauss wouldn't have thought of them as different, just longer.

M. It sort of brings up what a proof is *for* doesn't it?

S. What do you mean!? A proof is to show something is true!

- M. But those students found that doing measurements on their drawings showed them that ‘an angle in a semi-circle is a right angle’ was true, which you and I don’t consider a ‘proof’. What did they think of Anthea’s proof?
- S. Well, several of them had got essentially the same proof as hers - you know, by constructing the two isosceles triangles out of the right angled one - but her presentation was so short... Dan said that it was only because he’d used the same idea that he could follow.
- M. Are mathematical proofs and being convinced that a mathematical proposition is true different things?
- S. Seems like we’ll have to try to convince these students that mathematical proofs are more convincing *really* than their measuring demonstrations.
- M. That sounds a lot of a harder job than showing them how to do various sorts of proof. Not sure whether ‘teacher of meta-proof’ is in my job description! How’s Cabri going to help in that?
- S. Hmm ... they might have a better idea what’s true -
- M. - but won’t that demotivate them from doing proofs even more?
- S. I think it helps to have a firm grip on what is there -
- M. Like they’ll be there now! It’s nearly 8:30 already!

1.5.5 *Wis en zeker*³?

Friday 3:45 p.m.

- M. Have you done the fundamental theorem of calculus yet with your intermediate class?
- S. Next week, I should think. I’m having a bit of a struggle with the ‘you can’t have negative area’ line. It seems such a silly thing to get caught up on, but some of them are adamant that ‘negative area doesn’t exist’, it’s impossible, even if I wanted, to just say ‘shut up and swallow’.
- M. Is an integral an area then?
- S. Oh, don’t you start! What do you think it is then, a cheese sandwich?
- M. I mean, sure we can get answers to area problems by using integral calculus -
- S. Ah, not all, some 2D sets don’t have area!

³ From Freudenthal (1991) *Wis en zeker*, means ‘sure and certain’: “The Dutch term for mathematics was virtually coined by Simon Stevin (1548-1620): *Wiskunde*, the science of what is certain. *Wis en zeker*, sure and certain, is that which does not yield to any doubt; and *kunde* means knowledge, science, theory.” (p 1)

- M. That's sort of the problem, isn't it? Some 2D sets don't have area, and some integrals are not obviously 'areas'. So what has 'area' got to do with integrating? After all, 'area' is a physical thing.
- S. I think we're back to where we were on Monday, it's the count-the-squares approach to area that gets them started, like the perceptual beginning. Please don't tell them that some sets don't have area! Anyway, I bet you can just turn to a fancier sort of integration and you could talk about the area legitimately.
- M. But if you got into these fancy integrations, Lebesgue's or Borel's or whatever, is it the count-the-squares approach that will start you off, like it is for the Riemann integration that we teach?
- S. Is that what it's called?
- M. Think so. But these other integrations might give us a completely different set of answers to so-called area problems from the usual method. So what would be right?
- S. Sounds a bit like the non-Euclidean geometry problem to me: our immediate space seems to be Euclidean, but actually it's not. And people used to think the Euclidean model was the only thing that was geometry, but actually it's not.
- M. I think you've got two different 'actuals' there, actually! What physical space is ... well, that's an empirical problem. But what geometrical axiom systems 'work', as an abstract system, well, that's a pure mathematical problem, isn't it?
- S. But loads of maths comes from problems that are from the 'real world'. Newton developed the calculus for, what we would call, applied problems. It's only because those problems were solved with the help of his methods that it occupies such a vast amount of what beginning scientists and engineers are taught.
- M. Maths works!
- S. As long as you employ the right maths!
- M. But the maths you don't apply can still be true, it's just not real.
- S. I don't agree with saying that sort of maths is not real, it just might not have been successfully applied *yet*. That's one of the things about maths isn't it: pure mathematicians work away at an obscure theory and, then, some one uses it in an applied setting, and it seems wonderful and amazing. Matrices, prime numbers, fractals, and so on, have gone from pure to applied.
- M. So is all mathematics potentially applied?
- S. Sure, if scientists want to use a new form of counting, or whatever, what's to stop them?
- M. Nothing, if it works. But even if it doesn't work, as an application, it can still be true mathematics.
- S. So mathematics might rest on science, the people involved with science, to develop, and so what part of it that develops depends, essentially, on those people's interest-

M. or funding potential, that's why so much mathematics has been developed for military purposes-

S. yeah, so although science might inspire and use maths, it doesn't determine what mathematics is true or false.

M. We're back to square one! Does this help with the negative area problem?

S. I think so. I'll try this tack with them: we start off on a new topic as scientists employing maths we already know, like we were calculating the areas of trapezia, then a new idea comes to mind which encapsulates the abstractions of the investigation-

M. 'comes to mind'!! That is the biggest fudge I've ever heard!

S. What do you think, then?!

M. Oh, I agree with you! It just sounds like fudge that's all! Glad I'm a maths teacher not a philosopher, that's all I can say!

1.5.6 On the role of these dialogues

These dialogues illustrate themes which are investigated within the main body of the thesis. The dialogue format is intended to illustrate the way that philosophical issues do impinge onto the practice of mathematics teaching. The themes of the dialogues, belief, mathematical objects, notation, proof and *wis en zeker*, are not neatly assigned to specific chapters for individual discussion, but permeate the formulation of the sub-theses and inform the arguments therein.

Chapter 2: Methodology

2.1 Introduction

'Methodology' means 'the study of method', so writing about methodology involves writing about methods used for the investigation in question ('the investigation' being the content of this thesis); it is 'meta-method'. One can 'write about method' from several different perspectives (I am now writing about writing about method!). Possible perspectives, or meta-methods, include ethical, experiential and political, for example. The point is, that the content of the thesis depends on the perspective from which the author views the basic thesis question. The basic thesis question, here, is 'What is it to come to know mathematics?' and one would not expect a unique answer to so broad a question. This is why it is important to clarify the basic stance and explain what investigatory methods seem appropriate for that stance.

The basic stance that I shall take is 'experiential'. My interpretation of the meaning of having an experiential stance is part of the subject matter of this chapter. I shall also explain why I consider the most appropriate principal method of investigation - (given this stance and my particular starting question) - to be that of British-American philosophical analysis⁴ together with liberal exemplification with experiential items. Such a method could be termed 'analytic-experiential'. The reasons for adopting such a method, like developing the meaning of 'experiential stance', is also intended to unfold throughout this chapter. 'Experience' seems to feature twice in this statement: as meta-

⁴ This broad term connotes the English-language medium philosophers from Locke, Berkeley and Hume through to their descendants in this century like Russell, Quine, Putnam and Dummett. Arguably the category should acknowledge an Australasian contribution too, given the work of Armstrong and colleagues. A thesis could be written on whether this term does determine a coherent philosophical tradition. My purpose of using the term is to position my mode of enquiry away from that of the linguistic continental philosophers like Lacan, Derrida and Foucault. Why? Consider this analogy: Someone working hard to learn to play the piano cannot be expected to just pick up a violin and produce music. I was taught where the keys were in British-American philosophy many years ago. So when I wanted to make a philosophical sound I went back to an instrument on which I could produce some notes. I may appreciate

Chapter 2: Methodology

method, or stance, and as the source of examples to clarify the terms of the more general analysis. Is this legitimate? I believe so: experience is both a source and a check, but the argument is to be more general and, therefore, potentially applicable beyond my personal experience.

2.2 A student's progress?

Before the philosophical analysis can begin, using the stance I have declared, I must give an account of the experiences which gave rise to the appropriation of philosophy-plus-examples as a method.

2.2.1 Initial experience of research

For two years, during the late eighties, I worked half time at Oxford University's Department of Educational Studies, (OUDES), while continuing with my permanent school teaching job at The Cherwell School. The first shoots of this thesis work can be traced back to experiences at OUDES, though the roots go back to the interests in mathematics and philosophy I had as a teenager.

At OUDES, I initially misconstrued the job of a 'mathematics educationalist' because I did not realise that producing educational research was of utmost importance to being a successful professional in this area. This was hardly surprising, for I had been employed principally as a PGCE tutor who was in touch, by virtue of my other employment, with mathematics teaching in comprehensive schools. Nevertheless, I eventually got wind of the importance of research and publication and made a start on a small research project while employed at OUDES.

To get started on research, I talked informally with members of the department⁵ and decided to follow my long term fascination with the philosophy of mathematics.

the endeavours of others playing other instruments, but I have to be mindful of practical constraints, in trying to make a reasonable sound, and not try extend my skills too far.

⁵ Donald McIntyre was particularly generous with his time and constructive advice.

Chapter 2: Methodology

Questions that had puzzled me as a teenager - like the universality and applicability of mathematics - were still there to be investigated. It seemed that an educationalist's project could be conceived by interweaving something to do with learning, or sociology of schooling, or psychology, or curriculum design, etc. with the rather abstract philosophical interests I already had. I was starting to be able to envisage an investigation concerning some aspect of the nature of mathematics projected into the educational domain (to use a mathematical metaphor).

Practical considerations were paramount: any project I undertook had to be manageable, in terms of work load, and economical. Clearly, I should use resources my school could offer, the most significant of which were the students. So, this was my experience of starting on a research route:

- 1) declare an interest - mine was 'the nature of mathematics';
- 2) be opportunistic about resources - i.e. use what is available: my students;
- 3) ask a question related to the interest, an answer for which should yield from judicious use of the available resources - my question was 'What were my students' views on the nature of mathematics?';
- 4) report some of the findings - which I did in a professional (rather than academic) publication (Rodd, 1993).

This is a route typical of practitioner researchers as reported by Fletcher (Fletcher, 1993) and, indeed, echoes the advice given in Edwards and Talbot's first chapter (Edwards and Talbot, 1994, pp 3 -16). While *concepts within* the question included philosophical concepts, the *content of* the question was to do with peoples' attitudes. It was, broadly speaking, a social scientific question which I had approached using rudimentary social scientific methods; the tools used being questionnaire and interview.

2.2.2 *The next stage*

A few years later, I moved to a new job at St Martin's College in Lancaster. I no longer had a supply of school students, but had adult students to work with (aged from 18 to 50+) on both mathematics and pre-service courses. I was, by now, registered as a prospective PhD student. As it is reasonable to 'adopt, adapt and improve' previous experience, I expected to work with my new students on their ideas about the nature of mathematics and coming to know mathematics as the core of my PhD thesis project.

But this was not to be, for several reasons, which I shall shortly describe. And the consequence was a fundamental change in method: from social science to philosophy.

The H.E students were not a similar resource to the school students with whom I had worked previously. My explanation for this is as follows: I did not teach these students over a long period but saw them for a term (or less) at a time, in order to teach a specific module of their course. I did not know them as well as individuals as I had known my school students. This resulted in my not feeling comfortable enough to organise them, individually, as subjects for a further investigation of the type I had done at Cherwell.

What I did try was to ask the members of some of those classes which I taught mathematics for written reflective comments, to see if some texture of their views of mathematics could be discerned in this way. I found the results of this approach very disappointing: the students were either glib or scathing and they did not see reflective, post-session writing as a worthwhile activity. They responded fine as mathematics students, but they did not seem to see the worth of pondering the nature of the subject that they were preparing to teach.

Of course, 'I did it wrong'. This is not intended as a negative comment towards the students or on my relation with them, but a statement of my appraisal of my skill as a social scientist-cum-mathematics teacher. I believe, now, that it should have been possible to get interesting, provocative responses from these otherwise quite adequate students. But I have not invested the time in developing a social scientists' skills, so I

cannot knowledgeably say what a correct method would consist of; the belief remains an untested conjecture. At that time, I did not have the where-with-all to seek to improve my classroom-based research techniques: my experience included (what I now interpret as) not perceiving a lack of skill in technique. The consequence of my at-that-time-interpretation of my experience with these students, was that I did not expect to progress in my research by attending to students *said* about the nature of mathematics. The contents of this tortuous paragraph are intrinsic to my account of experience, because they constitute methodological remarks, i.e. they are supposed to explain why I used certain methods not others.

My next move was to look at what students *did*. Not in terms of gestures or other social interaction (like Arcavi, 1994) but of what their written mathematics consisted. After all, I had to mark their coursework and projects, I might as well consider these as data. I was being 'opportunistic about resources', as before.

I presented part of this work at a seminar⁶ of mathematics educators. From this I learnt something about the social dynamic of seminars: For example, the participants in this seminar liked seeing students' work; they liked to talk about it and construe meanings from it. A consequence of this interpretation of my experience was that questions which involved explicit philosophical analysis were not, in that context, made apparent to me; I am not saying that such questions were not raised.

At this seminar, I presented a construct which I termed 'mathematical moment'. A 'mathematical moment' was to indicate a point of transition for the learner from not-knowing to knowing. This was supposed to be an analogy in developing mathematical practice to a 'critical incident', as used in developing teaching practice (Lerman, 1994). To illustrate this idea I presented data (a fragment of dialogue and written mathematical workings) from a student, Gina, who had been working on the problem of proving that opposite angles in a cyclic quadrilateral added to half a turn. What was I asserting about

⁶ The Open University Centre for Mathematics Education research student seminar, 4/2/95.

mathematical knowledge here? I was making the bold claim that there had been a sort of 'change of state' for Gina before which she did not know the proposition, after which she did. Dick Tahta, in his wisdom, suggested that I go back to Gina in a few weeks and talk to her about her knowledge of this particular geometric proposition, which I did. In a subsequent brief, informal chat⁷ after class, Gina told me that she did not feel that she 'knew' the proposition - although she had 'responded correctly' when asked the fact of the angle relationship. I asked her to prove the proposition and she got stuck. With a nudge she wrote down a proof, but she still did not want to assert that she 'knew' the proposition!

This forced my awareness of the validity of such a construct as 'a mathematical moment or transition'. Despite my school-teacher experience of talking with colleagues about students' getting it' (or not!), as an aspect of an epistemological theory (i.e. a theory of knowledge), I was not offering a theoretical justification. Furthermore - and here I return to the notion of social science technique - I considered that, no sort of scrutiny of what my students did, or rather seemed to do from observation of their actions and written work, could justify it either. This feeling was analogous to my belief that it was not possible to gain quality information from what students said.

2.2.3 A change of method after reflection on earlier work

In other words, I was faced with a dilemma - either develop technical social science research technique or change method of enquiry - and, at that time, only horn of this

⁷ To my question "Can you say anything about the angles?" She replied instantly "Opposite angles add to 180degrees". To my "How could you show that?" she paused, suggested constructing the quadrilateral's diagonals [this was the construction that she'd started with those weeks ago]. She paused again and then silently drew in 2 radii, 'angle at the centre is twice angle at circumference' diagram with q and $2q$ marked. Pause again. She then mumbled something about isosceles triangles and joined the centre to the 'top' vertex. She originally interpreted this new constructed radius as bisecting the angle q , but a prompt to reflect dislodged that notion, and she labelled the base angles of the isosceles triangles x and y as shown in the diagram. Another pause; "I think there is another triangle" When the final radius is drawn in (after fumbling a bit with the labelling), she writes down *exactly* as she did six weeks previously: $2y+2b+2a+2x = 360$. Then, unlike before, she returned to her diagram to look, then back to her equation. Then, as before, without obvious anticipation or foresight, she wrote " $y+b+a+x=180$ ", and then said, while looking hard at the paper "That's it, that's it". I then asked her if she felt she knew this now. She replied, without hesitation "No, I don't feel I know it".

dilemma which seemed possible for me to take was to change the method of enquiry. The basic question remained, 'what is it to come to know mathematics?', but now I was not going to investigate that question through my students attitudes to the nature of mathematics, but to approach the question on a purely philosophical level. Why I thought that philosophical technique would be possible for me where social science technique was not, comes down, I conjecture, to my having studied philosophy as a minor subject as an undergraduate. In retrospect, I think I was optimistic! Nevertheless, an important point about research can be construed from these remarks: a teacher is a phenomenon of social science, not a social scientist; a teacher may be interested in philosophical questions, but that does not make her a philosopher. My experience with my H.E. students had prompted a change of method. Any further investigation was going to be conceptual rather than empirical.

My next task was to try to understand how educational research could also be philosophical research.

2.3 Educational research and philosophy

Although I had interests in philosophy that I wanted to pursue, and some undergraduate philosophy courses to my credit, my expertise and experience was in teaching school (and some early H.E.) mathematics. Philosophy of mathematics is a two and a half thousand year old field and, at the stage about which I have just been talking, I was aware only of parts of Plato's work and some ideas from the British empiricists; I had an unanalysed attraction to Kant's notion of the 'synthetic a priori' and an idea that the formalism, constructed in the early twentieth century, had crumbled with Gödel's theorem. These rusty undergraduate ideas did not seem to help me with my investigation on 'what is it to come to know mathematics'. Hence, I still felt that I was engaged in some sort of educational research, albeit now with a philosophical method - whatever that might be in practice.

There was a tension, at this stage, between the people-orientation which I thought was intrinsic to an educational enquiry and the abstraction, or lack of person-orientation, I

associated with philosophical enquiry. I sum up this tension in terms of the questions: Is educational research necessarily social scientific? Does philosophical method have to abstract away from experience? I do not think either of these questions needs to be answered in the affirmative:

2.3.1 On social science and educational research

In the introduction to his 1992 research guide, Hammersley states: "Educational research is a very wide field, and one whose boundaries are not at all clear, it merges into other areas of social and psychological research.", (Hammersley, 1992, p 4). In particular, I was aware that there was work of a social scientific nature related to mathematics in education: e.g. Bishop (1988) worked ethnographically as did Jaworski, (1994). By contrast, Hart, (1981), worked with statistical analysis on large samples.

One of the key issues in social scientific research is that of the role of the researcher/observer in the production of his/her thesis. For example, if a teacher asks her pupil if he likes mathematics, how many different ways can we interpret his answer? He might answer 'yes' because he does not want to feel awkward, or might answer 'no' because he is within earshot of other pupils to whom he does not want to appear a 'swot'. The approach advocated in Hammersley's guide is that "evidence used by researchers is systematically recorded and open to public scrutiny. This evidence may come from many different sources... [including] written responses by subjects..., a researcher's detailed notes, or even audio- or video-recordings." (ibid. p 30). In this way the source of an interpretation may be available to others, and while no ultimate objectivity is claimed, there is an honesty about how the social scientific theory was construed. In the scenario above, a scrutineer might query the abruptness of the question asked or the situation in which the child was expected to respond genuinely.

My work on student attitudes or responses to mathematics was essentially social scientific in approach. They were preliminary studies which could have been developed, as I have alluded, with further technique, to sit quite nicely in the sort of social science research that includes education. Indeed, considerable work has already been done in the area of

student and teacher attitudes, for example by Alba Thompson (Thompson, 1984) and Raffaella Borasi (1992).

Returning to the first of the questions posed in the second paragraph of 2.3, educational research has a social scientific branch, it does not follow that educational research is all social science. Using Hammersley's conception, social science is characterised by the methods used, as briefly discussed. It would, therefore seem possible to employ a different method to address questions which, hitherto may have been approached in a social scientific fashion. For example, the philosophy of mathematics is not only an historical phenomenon but also a contemporary academic discipline, which includes epistemology and ontology as well as formal logic. Could the methods used in contemporary philosophy be used within my broadly educational investigation? I hope that this thesis exemplifies that the answer to that question is 'yes'! But before I consider what sort of methods might be appropriate, such that - apropos the second question posed earlier - philosophical method does not have to abstract away from experience entirely, I need to review different meanings of the word 'philosophy'.

2.3.2 Some meanings of 'philosophy'

In common parlance, the word 'philosophy' may be used synonymously with 'attitude' or 'view', for example, to the question "What's your philosophy of life?" one might reply "Oh, live and let live". This attitudinal sense can be characterised by the following: the teacher whose attitude/philosophy guides her to a practice of offering 'real life problems' to her pupils might be attributed as holding a philosophy/attitude of mathematics that is characterised by 'mathematics is a culturally specific problem-solving tool'. I want to avoid this attitudinal sense of the word 'philosophy'. For it is perfectly logical to hypothesise an individual teacher who works with 'real life problems' as a pedagogical device and whose philosophy/attitude to mathematics is characterised by 'mathematics is the one universal discipline where absolute truth is obtainable' - which is contrary to the view espoused above. Whether there is a statistically significant correlation between a particular viewpoint and particular teaching style may or may not be the case - such analysis is social scientific again and outside my defined domain of interest in this thesis.

There is substantial literature on this attitudinal domain of enquiry (e.g., Jaworski, 1994) which explore the notions of consequences of teachers' views of mathematics in their classrooms.

A second, quite different use of the word 'philosophy' as it pertains to the philosophy of mathematics, is to specify a choice of philosophy from the received categories of logicism, formalism or intuitionism. A discussion of 'philosophy' in this sense requires an understanding of these philosophical schools which were influential during the first two-thirds of this century. My first study, as I have related, was to explore with which of the philosophies of mathematics, logicism, formalism and intuitionism, students identified the nature of mathematics. For that research I made up some statements which I considered characteristic of each those standard philosophical positions respectively, then I tested students' views against these, using questionnaire and semi-structured interviews. (The detail of the method was given in Rodd 1994, while my overall interpretation is in Rodd, 1993).

A third meaning of 'philosophy' connotes an epistemic view. An epistemic view is the attitude a person has to how knowledge is obtained. Specifically, a belief about how mathematical knowledge is obtained must, also, include some notion of what mathematical knowledge consists. So an epistemic view of mathematics implicitly includes an ontological view. For example, one of the results from my work with Cherwell students was that, despite often having the same teacher, students held quite different epistemic views. For example, Jeannie and Alex had been educated together since the age of nine, but Jeannie seemed to conceptualise mathematics as formal, absolutist discipline, outside science, and access to that knowledge was via an authority (teacher). Whereas Alex's view could be interpreted as conceptualising mathematics as essentially scientific and fallible. Alex considered access to that knowledge was via experimentation. This means that such students believed that mathematics was a different enterprise, both in its nature and as an activity. Ruthven and Coe's research into GCE A level students' views on mathematics reported more detailed, but substantially similar, conclusions to mine, (Ruthven and Coe 1994).

A fourth use of the word 'philosophy' occurs in the phrase 'philosophical method'. I am not asserting that we can divorce philosophical methods from philosophical problems. Nevertheless, I think that it is worth trying to isolate the features of the philosophical method. In Glymour's (1992) philosophy textbook, he opens with:

"Philosophy is concerned with very general questions about the structure of the world, with how we can best acquire knowledge of the world, and with how we should act in the world." (Glymour, 1992, p 3)

These branches of enquiry are known, respectively, as metaphysics, epistemology and ethics. The first two of these are relevant to my question about coming to know mathematics. Glymour grasps the nettle of philosophical purpose by asking rhetorically: "Isn't the question of the structure of the world about *physics*? Aren't questions about how we acquire knowledge and about our minds part of *psychology*?" (ibid. p 3). He then lists questions that are "some how too fundamental [to be answered] by a planned program of observations or experiments" (ibid. p 4).

So, for example, the notion of infinity cannot be experienced in a direct way, as roundness and seventeen-ness arguably can. What then is the nature of such a 'mathematical object' (the scare quotes indicate that I am trying to avoid begging the question!) and how is its nature known? These are concerns about ontology and epistemology, which are not of the same *type* as those reported in my 1993 article. The concerns in that article were to do with students' attitudes to mathematics learning and their epistemic views, not with mathematics itself and routes to access mathematical knowledge.

A final sense of the word 'philosophy' has a derogatory sense when used in phrases as 'She's just philosophising!'. The connotation is that the speaker is not grounded in practical matters but giving forth on generalities that have no particular relevance. One of the purposes of using experience as a check (2.1) is to avoid 'just philosophising'.

2.3.3 From first exposure to current use of 'philosophy'

I shall trace, briefly, some stages in my understanding of the term 'philosophy'. I was first introduced to the term 'philosophy' by my Latin teacher, when I was about 15. I cannot remember the details of her explanation, but the sense of 'philosophy' I received was that philosophy was concerned with general and unbounded ideas like 'knowledge' and 'truth'. I got a similar sense from the recent novel/short course in Western philosophy, 'Sophie's World' when Sophie's 'teacher' writes "the only thing we require to be good philosophers is a sense of wonder" (Gaarder, 1995, p14).

The domain of philosophical enquiry is immense and concerns itself with fundamental issues of existence and knowledge of existence as well as, for example, issues concerning ethics or the mind. The first book on philosophy I started to read was 'The Problems of Philosophy' by one B. Russell, (Russell, 1912). I remember picking it up, by chance, in the book shop next to the bus station on my way home from school. A Miss Barbara Russell was my history teacher at the time and I thought she might have written the slim volume, but by the time I realised who the true author was I was pondering the nature of the reality of objects such as tables and chairs. I came to philosophy from the metaphysical direction, interested in questions of existence, and the subsequent epistemological questions of our knowledge of that which exists. Curiously, this is still the centre of my philosophical interest, and the sense which I shall use the term 'philosophy'. In this thesis I focus my attention to 'mathematical existence' and 'knowledge of mathematics'; thus delineated we have the classical distinction of ontology and epistemology: what there is and what is known.

Returning to the general theme of this section, (2.3), the key point is to justify my turning to a philosophical method and to give a brief indication of this method's characteristics. The reasons why I changed my approach were, of course, a function of the research process and a result of the research itself.

The move towards the more philosophical enquiry can also be traced by noting the changes in emphasis in research questions: My focus changed from 'What do my students

Chapter 2: Methodology

consider mathematics to be?', to a question less dependent upon entering the minds of others, like the question I have frequently quoted 'what is it to come to know mathematics?'. With this firmer grasp of what 'philosophy' meant, I had the challenge of marrying this method with an 'experiential' stance.

My approach to this methodological challenge was to incorporate *practitioner* research with philosophical analysis. In order to explain how this unlikely amalgam could constitute a research method in mathematics education, this next section attempts to explain what I see as the important features of practitioner research and its relationship with academic British-American philosophy.

2.4 *Practitioner research and philosophy*

My aim was to work with philosophical questions (in Glymour's sense) about mathematics and existence-in-learning, yet not become detached. This may seem paradoxical. How can I combine nitty-gritty details with abstractions? My practice as a mathematics teacher certainly involves immersion in mathematical and classroom detail. On the other hand, philosophical questions probe fundamentals which empirical enquiry, per se, will not satisfy. I want to link the content of philosophy and mathematics-in-education in this thesis whilst retaining my 'experiential' stance.

2.4.1 *Practitioner research*

Practitioner research constitutes a broad research methodology that encompasses many methods and disciplines. It is a 'methodology' because it rationalises methods of enquiry. The key to that rationalisation is the observation that a practitioner has insights into the phenomena of interest by virtue of their being immersed in that work; e.g. they are part of the phenomenon that is their classroom. That very immersion moulds the practitioner psychologically, socially and culturally in quite unquantifiable ways. The recognition that it would be impossible to distinguish all the variables that constitute 'this teacher' and research his or her practice from the 'outside', leads to the distinctive practitioner research methodology: The responsibility for making sense of and communicating

Chapter 2: Methodology

practice is taken on by the practitioner who researches the phenomena of interest “from the inside” (Mason, 1993, p 1).

Although this sort of research may be introspective, oftentimes the research does involve collection of data. For example, a teacher questioning the relative performance of girls and boys might use some quantitative analysis on data from the whole school records or s/he might interview a sample of children throughout the school. Either of these methods, if executed to a standard acceptable to the teacher’s peers, governing body or examiners, are within the practitioner research umbrella, and either might effect changes in that teacher’s perception of related issues, for example her consciousness of class-discrimination may have been (unexpectedly) raised. In their guide to practitioner research, Edwards and Talbot state:

“Any piece of research carried out by a practitioner which has as its focus the concerns of that practitioner’s profession can be defined as practitioner research.” (Edwards and Talbot, 1994, p 52).

Furthermore, action research is a special case of practitioner research in which practitioners “engage in researching, through structured self-reflection, aspects of their own practice as they engage in that practice.” (ibid. p 52). Action research, then, according to Edwards and Talbot, is the sub-category of practitioner research in which changes in practice are *overtly* sought. However, I would argue that any serious study of one’s work is likely to produce a heightened awareness about the issues involved, and so will, despite what the original plan might have indicated, modify practice. So, no clear distinction can be made between practitioner research in general and action research in particular when consideration of the practitioner’s awareness is taken into account. (See, for example, Mason, 1993).

In order to participate in practitioner research a ‘practice’ is required; this practice provides the experiential basis for the reflective analysis integral to the practitioner’s project. The kind of practices which Edwards and Talbot’s work is intended to support are principally those of educational, health and social service professionals. My practice

Chapter 2: Methodology

as an educational professional was well established. In order to link philosophy and mathematics teaching, from an experiential stance, my aim was to establish a 'practice' of philosophy.

2.4.2 *Philosophical research*

What is 'philosophical research'? In a limited sense, like any other practice, philosophical research is part of the job of a professional in that field. In a broad sense, it is the enquiry of anyone seeking the answer to essence-questions, as Glymour described. So roughly, there is a 'professional' dimension of 'philosophical research' which stretches from 'academic' to 'ordinary'. Another, transverse, space of 'philosophical research' is given by the *method* by which answers to essence-questions are sought.

Philosophers, in the broad sense of the word, try to convince those interested in their solutions to essence-questions through various means, for example: discussion with questioning, story, example, threat, argument. Mystical insight, as well as logical deduction from specified premises, can constitute a rationale for accepting a proposition about the nature of things. The communication of this insight may well be through a story or threat. An 'argument', in this wide sense of possible methods of conviction, is one form of conviction-method among many. An argument is characterised by its specification of assumptions, use of some standard rules of inference, and consequential assertion of a 'proved' proposition. Mathematics and philosophy are both disciplines which use such methods to convince. But the style of an 'argument' can take quite different forms. Before delving into argument style, I want to suggest some dimensions of philosophical method.

Academic philosophy's paradigm method is based on logic, on ancient *forms* of deduction. Scientific enquiry likes to employ logical methods too because of the potential transparency of logical forms of reasoning. Scientific enquiry involves empirical evidence for the conclusions it purports, which involves specific observational data related to the research question. Philosophical analysis has included, during this century, vast tracts of symbolic logic, which did not involve itself with specific data and the comprehension of

which was for the few. Russell and Whitehead's 'Principia' was one of these a-empirical, formal works. Clearly, such work is on the 'academic' end of the philosophical research dimension mentioned above, which I called 'professional'. It is also on an extreme of another two discernible dimensions of philosophical research: 'experiential' and 'linguistic': experienced 'forms of life' are not involved and language has been boiled down to its logical structure. To contrast with a Russell-Whitehead approach, many of the easily-accessible novels of Iris Murdoch are philosophical in the academic-professional sense, but they employ quite different methods of conviction. Her works are first stories. In terms of the linguistic dimension of philosophical method, ideas about essence-questions are communicated through narrative rather than through deductions, (although a device she uses sometimes is to have a character present a logic-type argument). In terms of the experiential dimension of philosophical method, Iris Murdoch uses characters to give her reader insight to forms of life (like solipsist, sophist, empathist, spiritualist, embodied by these characters), which is on an opposite extreme to the sense datum type observation of a scientist.

The philosophical research which I have been trying to do here, in terms of these dimensions, is 'academic' rather than 'ordinary'. This is because of the 'professional' function this particular piece of work serves. But the positioning of this research on the other dimensions is less clear cut. This is because the philosophical work is subject to my mathematics teacher's practice, which includes working with people and working mathematically. So I have to attend to 'forms of life', like those of different human cultures as well as to scientific data, on the 'experiential' dimension. And I have to give discursive windows on these forms of life, as well as make deductions with logical transparency on the 'linguistic' dimension. This outlines the 'philosophical practice' which I have aimed to pursue.

In more detail, the philosophical reasoning which I aim to employ has the character which Moulton (1983) describes as *enchelus*. This is "a method of discussion frequently identified with the Socratic method ... its success depends on convincing the other person, not showing their views are wrong to others" (p 156). Moulton's development of *enchelus*

Chapter 2: Methodology

is designed to counter the 'adversary paradigm' which, she observes, is prevalent in academic 'philosophy reasoning'.

“Under the Adversary Paradigm, it is assumed that the only, or at any rate, the **best**, way of evaluating work in philosophy is to subject it to the strongest or most **extreme** opposition. And it is assumed that the best way of presenting work in philosophy is to address it to an imagined opponent and muster all the evidence one can to **support** it. ... conditions of hostility are not likely to elicit the best reasoning. But when it dominates the methodology and evaluation of philosophy, it restricts and misrepresents what philosophic reasoning is.” (p 153)

A particular consequence of encheleus argument is that artificial counterexamples are less detrimental to the validity of the ideas than in the adversarial mode. The argument respects logical form, but can move along the 'linguistic' dimension I have described to give a narrative insight into the 'experiential' dimension, from a form of life to an observation.

This expansion of philosophical practice is intended to specify the method of enquiry I use in the following chapters. The questions I ask about mathematics and coming to have knowledge of mathematics are fairly academic in the professional sense. For the most part they are traditional questions from the philosophy of mathematics. I hope to offer some novel insight into these questions because of the experiential stance I take. Specifically, because this work is practitioner research from a mathematics teacher, there are mathematical and teaching discourses which I can use for the linguistic dimension of this philosophy reasoning. The mathematical discourse includes some logical forms which are shared with academic philosophical discourse: philosophical enquiry recognises mathematics as a domain of interest, stimulating questions for research on its nature. In terms of the experiential dimension, a mathematics teacher's practitioner research has access to mathematical science as well as to students lives. Knowing (some) mathematics

is part of my life, (as exemplified in chapter 3), as is being involved with my students when they learn mathematics⁸.

2.4.3 Considerations about 'validity'

The meaning of 'validity' has different connotations whether a practitioner-researcher or philosophical discourse is employed. In the former case, there are two discernible branches of the research which are assessed for validity. These are how the data are gathered, (which includes whether these are relevant data for the research question), and the method of data analysis (Edwards and Talbot, op. cit., pp70-7); a questionnaire is not valid if it is biased and, assuming a sound questionnaire, the analysis is not valid if, say, statistical analysis contains gross calculation errors. In the latter case, 'validity' can also be used in two ways. Firstly, the root concept of philosophical validity comes from privileging certain forms of reasoning. A syllogism, in *Barbara*, for example, is a valid irrespective of its components (Glymour, op. cit., pp49-52). The form is arbiter of the validity. Secondly, clearly related to the first meaning, is a wider sense of validity at the heart of philosophy reasoning. This asks of a philosopher's attempt to make a case for the truth of a proposition: Does this form of words constitute a valid argument? In order to be a convincing argument, the premises should be true as well as the deductions of a correct form. In practice, this can be very difficult. Interesting issues often turn out to involve many variables, or premises, and these premises are expressed in language which is open to interpretation⁹.

⁸ This work can be interpreted as phenomenological in the following way: "The sense of phenomenological statements is very much like that of an explorer's statements, for the meaning of both is similarly twofold: in so far as they claim to be descriptions of the 'land', they are at once epistemic (knowledge-claims concerning the land itself) and communicative (that is, invitations and guides intended to enable others to know what to look for)." (Zander, 1970, p 36) A thorough phenomenological interpretation requires a very careful use of what phenomenology means. Zander's idea that "everything can be, in some respects, open to phenomenological study" (ibid. p31) seemed too wide to be used as a method which would provide validation.

⁹ In chapter 6, Benacerraf's syllogistic argument for the impossibility of mathematical objects is discussed. It serves as a good example of a philosophical argument where the form is easily recognised as valid, but

Chapter 2: Methodology

In this thesis, the validity does not rest on the practitioner researcher model. This is simply because the case I make for the propositions I want to assert is not based on a data-gathering method. I make my case for the propositions by trying to unravel the meanings and consequences of important terms; for this my practitioner experience is crucial and use of examples integral. Furthermore, much of the thesis is concerned with applying philosophical concepts and distinctions to matters in mathematics in education. The issue of validity here becomes even less well-defined: whether I have used philosophical concepts ‘correctly’ is one of interpretation. For example, is my use of, say, ‘ontological commitment’ (developed in chapter 6) a ‘valid’ application of a philosophical concept in an educational domain? In ordinary language, ‘valid’ means acceptable, often with the sense of allowing permission. I have a valid UK passport, but not a valid visa for China. So a qualificatory thesis is also valid, in an ordinary language sense, if it is acceptable and serves as permission to proceed.

2.5 *Summary*

To summarise this methodological analysis: I started with an interest, arising from mathematics teaching practice, which included some philosophical questions. In the early stages of my research, philosophy had originally been a question of received ‘positions’, like the mind-independence or otherwise of mathematics, and the investigation was concerned with which of my students held which position. This was an investigation of a social phenomenon, but it was not really what I wanted to work on. The concepts fundamental to my interest were about ontology and epistemology and social science methods were not suitable for investigating questions about these philosophical concepts. This lead me to want to employ a philosophical method to research the questions I had about coming to know what mathematical reasoning is, what mathematics is about and what mathematical action could be. I noted that educational research does not have to be

where the validity of the argument rests on the interpretation, and subsequent assessment of the truth values of the premises.

Chapter 2: Methodology

strictly social scientific with respect to method, in particular, philosophical methods may be used.

I then had to rationalise the potential conflict between using a philosophical method (which generally should not depend on the particular experience of the individual writing) and an education-inspired question (which is inevitably rooted in the experience of the writer). I have tried to bridge this gap by using the notion of 'practitioner research' and claiming practitioner status as philosopher and mathematician as well as mathematics teacher. Philosophy reasoning can include, in its experiential range, narratives from life as well as about observation, and in its linguistic scope, logic about action as well as logic within mathematics. Moulton's interpretation of *enchelus* permits a non-adversarial, philosophical argument which need not collapse if some nicety is constructed to counter a part of the argument logically. This gives the possibility of relating mathematics to learners' lives rather than abstract from the interesting teaching-maths-stuff so much that all one is left with is a logical structure but no person-centred narrative. But it is still philosophy for all that. This thesis is not just a story, its validity rests on argument.

3. Chapter 3: Phenomena

3.1 Introduction

In the previous chapter, I explained why a question arising from interest in philosophical questions about mathematics together with experience of teaching mathematics might be tackled using a philosophical approach. Doing mathematics, as well as teaching mathematics, is part of the practitioner's experience which underpins this research. Can any of these essential experiences be captured in a report? The aim of this chapter is to communicate the phenomena of the underlying practices by presenting some 'data'. These data consist of a selection of writings on 'experience'. The experiences described therein are relevant to my task of philosophically accounting for 'what it is to come to know' higher school mathematics.

The main body of the chapter consists of three unequal parts: personal accounts of mathematical experience in 3.2; reports of what my 'teacher's eye' sees as characteristic of the essentially mathematical within the higher school mathematics curriculum in 3.3; and a report of 'pedagogical content knowledge' in my practice as a mathematics teacher in 3.4.

The data presented in these three sections are of different types and each of the sections serves a slightly different function in my attempt to communicate the experiential background which I am coming from:

3.2 is a 'data set' from mathematical experience. It consists of mathematical arguments, together with reflections, reader-tasks and philosophical questions which were provoked by the experience of doing the 'sums' and thinking about the process of thinking mathematically. The purpose of presenting this data set is to help the reader tune into both my experiential context of doing mathematics and my (philosophical) abstractions from that context.

3.3 is a 'data set' from higher school mathematical content. The contents of this data set are four broad areas of the curriculum which I have described from a teacher's point of view. I have chosen to try to capture the nature of this level of mathematics by example, rather than giving a syllabus list, because I think it is a clearer way to gain an insight into working with this level of mathematics. The purpose of presenting this data set is to help the reader focus on the sort of mathematics I am analysing.

3.4 is a data set from teaching higher school mathematics. This data set consists in an analysis of pedagogical content knowledge illustrated by a report of a particular teaching episode. The purpose of presenting this is to mark the distinction between mathematical content and interpretation of that content; what I want to consider is not the pedagogical representation¹⁰, but the mathematics 'itself'.

3.2 Mathematical experience: personal accounts

More explicitly, for each of the four episodes, (indexed by n), I present a short account of the situation prompting mathematical experience, (3.2. n), as well as a brief observation of some related philosophical issues. I then try to justify why the presented story constitutes a datum (an account of mathematical experience), and (indeed) in doing so expand on the datum itself. There are two aspects to this justification/expansion: the mathematical evidence, (3.2. n .1) and the meta-mathematical reflection, (3.2. n .2). Part of the understanding of this justification consists in engaging in a task, (3.2. n .3), usually with a mathematical and a reflective component. Through the experience of working on this exercise, I hope the reader will have gained an insight into the substance of my discussion. In other words, I want to justify my jump from mathematics to meta-mathematics by eliciting, for the reader, a similar experiential context as my own. Finally, for each of these data, I sum up what I see to be the next stage of the philosophical work

¹⁰This is Shulman's term, which he used in his widely cited 1987 paper in which the term 'pedagogical content knowledge' was coined. Fischbein, 1987, uses the term 'models' for pedagogical representations like 'debts' for modelling negative numbers.

to be done, (3.2.n.4). The notes in these sections are reflections from the time of writing the mathematical experience account, 3.2.n.

3.2.1 *On angle trisection*

One maths lesson, when I was about twelve, had been on geometrical constructions with ruler and compasses. We were told, as a fact, that, while you could use ruler and compasses to bisect an angle, it was not possible to trisect an arbitrary angle. I remember clearly, even now, how I went home that night and worked on trisecting various angles, for surely what is true for halves should be true for thirds?¹¹ While I do not recall the constructions I used, I do remember the dejection I felt - I was at this for hours, I'm embarrassed to say! - because I could never get the measure of the three angle parts close enough to feel sure that I had found a method for trisecting angles. It may have been that I had found a procedure, which was obscured by my blunt pencil and imprecise measurements. Was the inaccuracy the result of my wobbling the instruments or an imperfect method? There was actually no way of knowing whether I had succeeded or not. This was mathematical experience in the sense that I knew that I had not been convincing; I was aware that no empirical work itself could confirm that I had a method which worked, yet some mathematical results can have physical instantiations.

3.2.1.1 *Mathematical evidence for non-trisection*

(a) The proof of the non-constructibility of the angle trisection is given in Herstein (1975, pp 230-1). Herstein's proof depends on the concept of an irreducible degree three polynomial over the rationals, together with the result that a length l is only constructible if l lies in a finite extension of the rational field of degree a power of two. The result is a nice application of the theory of abstract fields and their extensions.

¹¹ Wittgenstein recognises this 'sequential imperative': "You have this sentence 'I bisect this angle' and you form a similar expression: 'trisecting'. And so you ask, what about the sentence 'I trisect this angle?' You are lead on here by *sentences*." (Wittgenstein, 1976, p88, italics in original)

(b) Using Cabri-géomètre (or other dynamic geometry package) it is easy to construct a dynamic version of a school-girl 'trisection'. Being able to vary the angle-to-be trisected shows, in a more convincing manner than a few drawings, how the approximate readings are really approximate, rather than just the result of wobbly hands. With such a dynamic construction available, the empirical evidence that this construction does not trisect is very convincing. This is because the angles of the 'trisection' are, for general angles, different by about 10%. Although Cabri is only accurate to the nearest degree, 3 or 4 degrees out indicates non-equality, not instrument wobble. Nevertheless, this can only be done for one failed trisection at a time! Because of the impossibility of testing all such constructions, the result clearly depends on an abstract, all-encompassing, proof.

3.2.1.2 *Meta-mathematical reflections*

The question of whether any angle can be trisected or not is to do with whether or not an action is impossible; not just impossible for me - like doing a triple back somersault - but as impossible as, say, all triangles being isosceles is impossible. If one understands a statement of impossibility, then it is irrational to experiment to overturn that result. Understanding that it is impossible means no matter how good a method for approximately trisecting you find, you know it will not be exact - even if you devised an approximate method that always gave you to the nearest degree accurate answers, you *know* that you have not done a trisection. As Wittgenstein remarked of this understanding "[T]he importance of the proof that trisection is impossible is that it changes our ideas of trisection." (1976, p88)

I like proofs of impossibility; they seem to me to be magic in a particular mathematical way. It is paradigmatic 'brain defeats brawn'. As Kac and Ulman (1968) express it:

"The unique and peculiar character of mathematical reasoning is best exhibited in proofs of impossibility. When it is asserted that doubling the cube (i.e. constructing the cube root of 2 with a ruler and compasses) is impossible the statement does not merely refer to a temporary limitation of human ability to perform this feat. It goes

far beyond this, for it proclaims that never, no matter what, will anybody ever be able to construct the cube root of 2." (p26)

The mathematical existence alluded to in this example is the scope of universal statements in mathematics. Just to understand these statements will have the effect of not even trying to attempt certain actions.

Measurement can serve as a perceptual realisation of a new concept in teaching mathematics - so we introduce decimal places, measures of angles, mechanical relationships using, essentially, science (predictable aspects of the material world we inhabit). The trick for the maths teacher is to get her pupils to use their perceptual experience to establish a concept, mentally, that will not be overturned by subsequent experiences. This sounds a little like how to establish an a priori concept 'synthetically' (in Kant's sense).

The Pythagoreans despaired when the paradigm of the accurate answer being measurable was broken (see, for example, Dunmore, 1992, p214-5). Teaching square roots today, I have found that many pupils find the notion of the irrationality of root 2 difficult to deal with; the never ending number is a big step to take in your notions of what it is that numbers are. But, as Richard Brown (from SCAA¹²) said at the BCME conference (1995) "There comes a stage when ' $\sqrt{2} = 1.414$ ticked correct' is not appropriate." So the question of what is measurable and in what way, is a mathematical question. This is rather different from the observation that representation of measurement using mathematical notation relies on mathematical structure (like the decimal system, 'arbitrary' but regular). But the answer to 'what is the measurement of...?' itself, is not so mathematical, it will never be exactly known via the measurement itself. Mathematical precision is of a different type to scientific precision.

¹² School Curriculum Assessment Authority

Chapter 3: Phenomena

The word 'algorithm' to me connotes a 'machine followable' sequence of instructions that will inevitably arrive at the relevant answer. Algorithms form an important part of mathematics which have, in some circumstances, made the study of mathematics seem unimaginative and off-putting. As I see it, something of mathematics' objectivity is realised in its algorithms, and while the routine following of algorithms can be come dull, the in-*corp*-oration of those algorithms is part of mathematical knowing, (I develop these ideas further in Chapter 7). Furthermore, the invention or discovery of algorithms can be a most creative enterprise. That there are algorithms, which machines execute perfectly, says something about mathematical existence. I could not find an algorithmic procedure for trisecting the angle, but did I work mathematically? To some extent I did, for I knew that whatever I did with my ruler and compasses, I could not be sure that I had found a method!

3.2.1.3 Trisection: how close can you get?

Design a ruler and compasses way of nearly trisecting the angle and then working through a proof to show mathematically why it doesn't work. A related pedagogical question is whether such a proof might convince a pupil that this method does not work. can this be a way of helping a pupil see that mathematics is beyond measurement? (Ideas about the rationale for belief are developed in chapter 5).

3.2.1.4 For further philosophical investigation:

The following are notes in progress that I made after I wrote the account above:

- ◇ The role of perception and empirical work in mathematics: the experimental as inspirer or irrelevance. How do particular perceptual experiences firstly help establish meaning? and then become but exemplifications of that meaning? When do they impede meaning?
- ◇ How are understanding and action linked?

Chapter 3: Phenomena

- ◇ That there are paradigmatic, or essence holding, examples and ideas in mathematics. What characterises these special cases that seem to encapsulate a generality? How do they exert this influence?
- ◇ Objectivity as realisable in algorithms but not in processes, like symbolising, generalising, testing, etc.
- ◇ Whether 'a priori' is, in practice, an empty or useful concept.

As I have mentioned, I shall leave these unedited because some of the themes and questions become important to the development of the thesis.

3.2.2 The group exists: understanding isomorphism.

The 'understanding' occurred in Hung-Hsi Wu's algebra course, (Math H114A), at Berkeley. We started group theory from scratch by looking at various sets which had a group structure. In particular we looked at the multiplicative properties of the following three groups: the set of non-singular 2 by 2 matrices with coefficients in \mathbb{Z}_2 , the set of isometries of an equilateral triangle and S_3 , the set of permutations on three elements.

The concept of group took life from this one example, from this point in time. Prior to this lesson, I'd have said that a group was one of those sets of permutations, transformations or whatever with some multiplication. I knew how to test for group structure using the group axioms. Afterwards those very sets were but manifestations (representations, appearances, forms) of the group that existed abstractly. I found it exciting because it fulfilled the promise of, and was an explicit example of a member of, the intangible mathematical realm that had tempted me to continue studying mathematics.

It was the explicit isomorphisms that gave me 'access' to the abstract group. Or, in more enactivist terms, after Varela, Thompson and Rosch (1992), the isomorphisms were the perturbations that provoked my further sense making. It doesn't matter in what school's

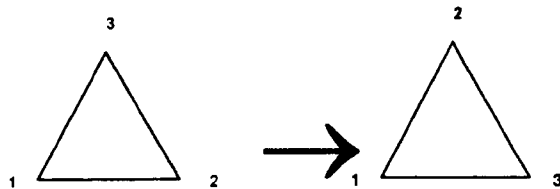
language this point is phrased, because, the point I want to make is: since that time, for me, abstract groups exist as mathematical objects. The abstract object, in this case the group, exists independently of any particular manifestation or representation of its structure. There is a sense of a 'Platonic object out there' and coming to know it, which can occur quickly, as if in a moment, when one is prepared.

3.2.2.1 *Mathematical evidence for abstract group existence*

On the isomorphism class:

Following standard notation, (see, for example, Herstein, 1975, pp 75-77), elements of S_3 can be written as (a, b, c) , where the digits a, b, c are 1,2,3 in any order. This set forms a group of 6 elements and is noncommutative.

Isometries of an equilateral triangle, also have the closure, associativity, inverse and identity properties required for group structure. Isometries can be specified by marking the change in position of the vertices relative to a standard position. The example shown is the reflection about the symmetry axis through vertex 1. Clearly, marking the vertices thus reduces these geometric transformations to permutations, the example being $(2,3)$ in Herstein's notation. The structure of these two groups coincides, but they had seemed to me very different at first.



How to see that the matrix set was also of this structure? First list its the elements and observe the associativity and closure using the Z_2 property $1+1=0 \pmod{2}$ and the non-zero determinant requirement of non-singularity:

Chapter 3: Phenomena

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, F_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, F_b = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, F_c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, R_1 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, R_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}. \quad \text{The}$$

identity matrix's function is clear, and hence inverses exist, but it is not as transparent that the multiplication of this group is the 'same as' the others.

The following Cayley table represents the multiplication structure of these matrices:

	I	R_1	R_2	F_a	F_b	F_c
I	I	R_1	R_2	F_a	F_b	F_c
R_1	R_1	R_2	I	F_c	F_a	F_b
R_2	R_2	I	R_1	F_b	F_c	F_a
F_a	F_a	F_b	F_c	I	R_1	R_2
F_b	F_b	F_c	F_a	R_2	I	R_1
F_c	F_c	F_a	F_b	R_1	R_2	I

Now, the notation has been carefully chosen to help elicit an identification between the matrix set and the set of permutations, which has already been seen to be interpretable as the set of equilateral triangle isometries. By identifying the reFlection transformations, given by the order two transpositions, as $F_a = (2,3)$, $F_b = (1,3)$, $F_c = (1,2)$ and the Rotation transformations, given by the 3-cycles as $R_1 = (2,3,1)$, $R_2 = (3,2,1)$, the multiplicative structure is obviously the same.

3.2.2.2 Meta-mathematical reflections

It is not often that one has the opportunity to pin-point a secure learning event. I think that the episode related was one of those events. It is difficult enough to try to capture these for one's self, and, of course even more uncheckable to attribute them to a student or pupil. Indeed, with regard to spotting secure learning in others, what do we have to go on? And how can this evidence be judged for its reliability? Every teacher does make judgements. Most students do (at some time) further their understanding.

This episode prompts discussion of the notions of instrumental and relational understanding, (Skemp, 1976). I would say that I had had some instrumental

understanding of a group prior to the lesson mentioned above. It would be tidy to say that my latter state of mind could be characterised as 'relational understanding'. But I don't think that this term properly captures the 'object like grasp' I recall obtaining. Indeed, at that time I had no further 'relational' meanings for 'group' - I did not know about different groups, or, for example uses of groups, as in physical theories, or about how group structure related to those of rings and fields, or what happened to groups under homomorphisms, or even that I was dealing with *the* non-abelian group of order 6. Although I was pretty ignorant, I had been able to apprehend objective abstract structure - in this case in the form of a mathematical object of the type named 'group', where groups are codified in terms of their four defining properties.

What the episode also illustrates is the use of paradigm examples as a pedagogical device. In this example of an isomorphism class, we find a structurally rich (the abstract group is non-abelian, and the given forms of it 'look' different) particular situation in which generality can be understood (as described, for example, in Mason *et al.*, 1982).

In the section 3.2.2.1 I have deliberately used a notational form to try to bring the reader closer to the sense of isomorphism that I experienced. So the geometric imagery of the rotations and reflections is embedded in the R and F notation for the matrices: does this make the acceptance of the structural identity too automatic, too easy? Was it not the sorting out of a notation that carried the promise of another interpretation the useful activity in terms of seeing the isomorphism? Being given the explicit Cayley table is probably not useful for someone learning about group structure, although the activity of laying it out themselves would force that person to work with the group elements and their multiplicative combinations, but perhaps (only) instrumentally.

What is it to see the same structure? Can it (just) be recognising the same notation? This raises questions for me about formalism as well as notation. Are the marks on paper the carriers of the 'truths'? or is notation a seductive screen on which is written a readable message? One of the faces of mathematics is that the symbolic medium is more than a set of inert logos (woolmarks, Nike's ticks and so on) but that "the symbol becomes symbolic" (Mason, 1980).

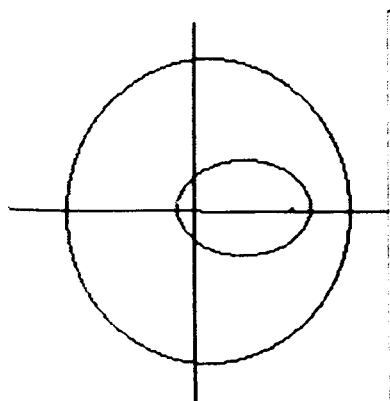
3.2.2.3 How to represent the structure so that the abstract form is recognisable?

Consider the group, (H, \wedge) , of order 8, where $H = \{ \pm 1, \pm i, \pm j, \pm k \}$ with the 'Hamiltonian' multiplication, \wedge , that can be construed geometrically as the vector product multiplication of the basis vectors, i, j, k , of real three dimensional space, R^3 . How do you come to know that (H, \wedge) is not $(Z_2 \times Z_2 \times Z_2, +)$? Which is a more obvious way of seeing the non-isomorphism of these two groups: a geometric representation or the structurally explicit Cayley table?

3.2.2.4 For further philosophical investigation.

- ◇ A change in being of mental state, and how to recognise it in self and as a teacher.
- ◇ More dimensions of understanding, or coming-to-know, than in the ideas of instrumental and relational understanding.
- ◇ The notion of the meaningfulness of the term 'mathematical object'.
- ◇ The role, recognition and pedagogical exploitation of paradigm, or essence carrying, examples.
- ◇ The power of formalism and the role of notation. The slipperiness and seductiveness of notation.

3.2.3 That looks like an ellipse



Chapter 3: Phenomena

Part of the students' work for my (2 year) PGCE geometry class was to teach part of the course. The week of 10/2/94, Brendan, Dan and Peter were introducing the LOCUS package to the group. Dan presented the CIRCLE program from the pack and showed how the program will plot the locus of the centre of a circle that passes through a fixed point inside a given circle and that touches the given circle. Brendan, smiling, shaking his head and looking towards me said: "The locus surely looks like an ellipse, but I can't see *why* it is". The problem then became my problem, both as teacher and 'mathematician'. The solution did not yield straight away, and I had to abandon a wrong track before I got out a proof that I present as correct. What is 'behind' the image on the screen? From a mathematical perceptions, mathematical questions are stimulated! Does proof confer truth? How else is conviction of a mathematical proposition secured? Does the text in 3.2.3.1 constitute a proof?

3.2.3.1 *Mathematical evidence for the locus being an ellipse*

Referring to co-ordinate axes sketched in the diagram, let the centre of the fixed circle be at the origin and of radius d . Without loss of generality, specify the fixed point, through which all the circles pass, as $C(c,0)$.

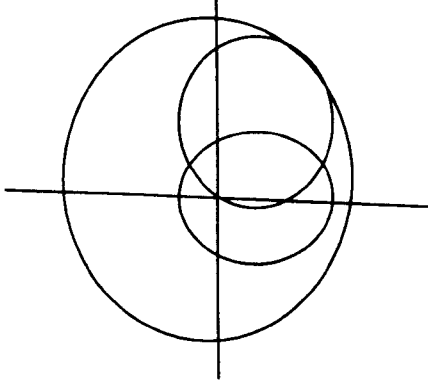
Now consider the cases of the largest and smallest circles: *if the locus were an ellipse*, the length of the major axis would be d , the radius of the given circle.

By symmetry of the hypothesised ellipse, the centre of the ellipse is at $x = \frac{c}{2}$. This is the x co-ordinate of the minor axis.

Then, by specialising to a circle, (one of two), with centre at $x = \frac{c}{2}$ we can find the length of the semi-minor axis, h , by setting up the equation for that circle. The y co-ordinate of the centre of that circle is h , and the radius-squared, by Pythagoras, is $\left(\frac{c}{2}\right)^2 + h^2$. This

gives the equation of the circle shown as: $\left(x - \frac{c}{2}\right)^2 + (y - h)^2 = \frac{c^2}{4} + h^2$.

Chapter 3: Phenomena



Now let $x=0$. Substituting in to the circle equation we get

$\left(0 - \frac{c}{2}\right)^2 + (y - h)^2 = \frac{c^2}{4} + h^2$, which yields $y = 2h$ or $y = 0$. This means that this circle, whose centre I'll call Q, passes through the centre of the fixed circle, O.

Let P be the point at which this circle touches the fixed circle.

Using this notation, OQ, QP and QC are all radii of this circle centre Q($c/2, h$), and equal to $\frac{d}{2}$.

This gives $h^2 = \frac{d^2}{4} - \frac{c^2}{4}$, as (0,0) is on this circle centre Q.

Therefore, the equation of the ellipse with constraints: major axis = d and semi-minor axis h is:

$$4 \frac{\left(x - \frac{c}{2}\right)^2}{d^2} + \frac{4y^2}{(d^2 - c^2)} = 1$$

Chapter 3: Phenomena

Using $a^2 = b^2(1 - e^2)$, and algebraic manipulation we get: $e = \frac{c}{d}$

The foci are at $F(ae,0)$ and $F'(-ae, 0)$ relative to co-ordinate axes coincident with the two lines of reflection symmetry of the hypothesised ellipse, which are $x = \frac{c}{2}$ and $y = 0$.

So, relative to these axes, we have: $ae = \frac{d}{2} \times \frac{c}{d} = \frac{c}{2}$, which is the fixed point through which all the circles pass! And the other focus is $\left(\frac{-c}{2}, 0\right)$, in the ellipse's axes co-ordinates or the origin $(0,0)$ in the old, which I shall now return to using.

Now, to show that the locus of Q is an ellipse I shall use the 'focal length property'; an equivalent definition of an ellipse.

Let the foci be at O and C , so it remains to show: $OQ + OC = d$.

Now, OQP is a straight line *as the circles touch*.

And $OP = d$, radius of the outer circle,

but $OP = OQ + QP$ and $QP = QC$, radii of the centre Q circle,

hence $OQ + QC = d$, and the locus of Q is an ellipse.

3.2.3.2 Meta-mathematical Reflections

The power of these sorts of computer simulations is that they give us a perception of a wide range of examples, where, otherwise, the very processes of drawing specific cases would mean that the scrutinisable set of examples would be very small. Not only would we have few examples to scrutinise, but the continuity between one and another would not be as apparent as on a computer simulation. The abundance of empirical experience prompts conjectures about the phenomenon observed after a shorter period of time than one would expect of an empirical conjecture without this tool. There needs, then, to be

less commitment by the individual to the problem before conjectures are asserted; this is one of the reasons why such IT is useful in teaching - students' imaginations can be captured more easily: the 'that' of the situation is more readily apparent. But, of course, this 'easy access' can mislead, if the cues are not well understood. For example, zooming in to $x\sin(1/x)$ at $x = 0$ for long enough can give a screen picture that appears to be locally smooth. Also, this 'easy access' does not mean there is not other, perhaps hard, mathematical work to be done, for which the students might not be so easily motivated.

In this case, I was not convinced that the locus was an ellipse from the screen alone - could it be slightly egg-shaped, for example? Personally, I was motivated to find/make up an explanation from which (a) I could be sure that it either was or was not an ellipse (b) could convince my students. My intuition was not strong enough to publicly assert: 'that is an ellipse' after seeing the screen images alone.

I know that I am more secure about whether a proposition is true if I have a symbolic representation of the phenomenon in question that has yielded a result through quite formal manipulations. Perhaps this explains why the screen images were not satisfying as 'answers'; as for Brendan, they functioned as a conjecture making stimulus for me. But other students couldn't see what the fuss was about - the shape was evident to their eyes. Now, we can all agree that our eyes deceive us often, and while those other students could, for this reason, see the *theoretical* point of a proof (of the locus being an ellipse) their cognitive frameworks were such that the practical question had been solved visually and the rigours of a symbolic proof were the sort of tedious game that gives mathematics a bad press. Or were their intuitions better than Brendan's or mine?

The shape of this proof merits some comment, for it is strongly influenced by the ellipse hypothesis. It is rather backwards to assert 'well, if it were to be an ellipse, then it would have such and such properties'. However, I think the proof works, because once I know those two special points $(0,0)$ and $(c/2,0)$ I can 'attach a piece of string' to them and, in the standard way, draw some ellipse. That this ellipse was the required locus depended on the crucial condition of the original construction that the circles touched. Then the

Chapter 3: Phenomena

implication that OQP was a straight line, although it was only used at the penultimate point, was enough to prove the theorem.

I found that the eccentricity of the hypothesised ellipse was c/d . This fact was not something I had an intuition about or was working towards showing, it fell from symbolic manipulations. But this was a result, not just another piece of algebraic noise: there are but two parameters in this situation, c and d with $c < d$ and as an ellipse has eccentricity < 1 , it was most satisfactory that $e = \frac{c}{d}$. I had no prior intuition of this result; it was intuitively right when it appeared!

3.2.3.3 *Algebra and geometry*

There is quite a lot of algebraic symbolism to wade through in the proof given above. Does this detract from 'seeing' the truth of the proposition in question? Is there a more geometric proof?

3.2.3.4 *For further philosophical investigation.*

- ◊ Mediating technological tools as perception enhancers.
- ◊ What does confer the truth of a proposition? Intuition as tested by structural proof and perceptual lived experience.
- ◊ The power of symbolic formalism to yield analytic truths.

3.2.4 *The Golden Circle property*

A part of the mathematics I enjoyed as a child was doing nice little Euclidean geometry proofs: there was a visual and deductive aspect which I found satisfying. At university we seemed to have out grown this type of geometry, and when I came to teach the subject in English secondary schools, the only geometry that students 'needed' to know were certain facts (like angle sum of triangle and angle in a semi-circle) but not their proofs, in the sense I had learnt all those years previously. The following little problem was prompted

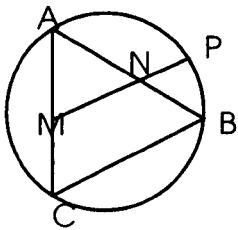
by reading an article by Coxeter in the Mathematical Intelligencer (Coxeter, 1994) and is illustrative of the deductive genre within ‘higher school mathematics’:

Given an equilateral triangle ABC inscribed in a circle. Let M and N be the mid points of AB and AC respectively. Produce NM to meet the circle at P. Find the ratio NM to MP.

The geometric configuration has properties which can be deduced independently of any diagrammatic representation (static or dynamic) by using rules of inference. Diagrammatic representation helps me to make sense of the problem and to have a ‘useful idea’ that opens the path to the deductive proof. But what interaction is there between my ‘playing’ with the problem and the ossified answer?

3.2.4.1 Mathematical evidence for the ‘golden circle property’

First, draw a diagram:



Produce NM to Q on the circle.

Now, $\triangle ANQ$ is similar to $\triangle PNB$, because:

$\angle QAB = \angle QPB$ (angles in the same segment QB)

$\angle AQP = \angle ABP$ (angles in the same segment AP),

and the triangles’ vertically opposite angles at N are equal.

As a consequence the ratio of sides have to be: $AN/PN = AQ/PB = NQ/NB$.

Manipulation gives me $AN.NB = QN.NP$

Chapter 3: Phenomena

Now, let the side of the equilateral triangle be of length 2.

$QN = 1 + X$. This is because $MN = 1$, ABC is an equilateral triangle and M and N are mid points.

Let $NP = X$. By symmetry $MQ = X$ too.

Form an equation: $1.1 = (1+X).X$

Solving the quadratic equation and taking the positive root, as we have a distance: $X = \frac{-1 + \sqrt{5}}{2}$. A representation of the Golden Ratio.

3.2.4.2 *Meta-mathematical reflections*

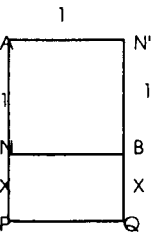
The mathematical problem is presented as a closed question, the very formulation of which suggests that there IS an invariant; there is a number that is THE answer. Hence, if I manage to get an answer, that number will either be right or wrong. I set off on the task acting as if there is something to find that is definite and predetermined. Whatever I shall do will not change the number; I have no causal interaction with the result. It is safe from my messing around whether I find it or not. It's a fact of the matter. (Sawyer, 1992)

How did I start on this problem? I used my diagram in an empirical way: I drew a large version of an initial sketch, then I sat and wondered and wandered about the situation. If I wanted to have estimated the required ratio, I could have done an accurate drawing or use a mediating tool, like Cabri-Géomètre (or similar). By doing some geometry experiments I might even conjecture a result which, because of the answer's theoretical irrationality, I could never actually measure exactly. I know that no representation *is* the configuration. My result here is not what I can see or measure but only what I can deduce from that which was given. After a false start, a pause, then a doodle leads me to symmetrise the diagram. So NM is now produced to Q on the circle. Now I see two similar triangles! A moment of insight, as I know that ratios come with this concept and a ratio is what is required.

Chapter 3: Phenomena

The relation $AN.NB = QN.NP$ is an expression of a theorem I should have recalled but didn't - I went back to the more ingrained notion of similar triangles. Ingrained not only because similarity is a concept used widely but also ingrained by visual sense.

The equation has another representation as the equality of the ratio of the sides in the two similar rectangles, $AN'QP$ and $PNBQ$:



There was something very confirming about the Golden ratio emerging as the answer. Why did it seem so improbable that this could be wrong? For sure, even an estimate by eye would indicate that it could be an approximate solution, but I had feeling of right-trackness, certainty, even. There was an aesthetic component to this experience which reinforced the existence of the answer which was held, structurally, in the quadratic equation.

3.2.4.3 A task: why is it a 'Golden egg'?

The graphic designer, Robert Dixon, in his book 'Mathographics', (Dixon, 1987), relates mathematics and aesthetics through elementary ruler and compasses constructions. One such construction that he shows completed, but without explicit construction lines, he calls 'The Golden Egg' (page 8) - there are several other egg constructions in the book. It

Chapter 3: Phenomena

would be interesting to observe whether, on working out how to construct the egg, you have the sense of the familiar ratio being uncovered, or otherwise.

3.2.4.4 *For further philosophical investigation*

- ◇ That mathematical results, may be conjectured, but are not established by empiricism, but by the paradigmatically mathematical method of proof which can both illuminate (enhance intuition) and yield necessary consequences.
- ◇ We develop ideas, new 'perceptions'/understandings/connections about the same, familiar thing.
- ◇ There are geometric facts of which we can have knowledge.
- ◇ Certainty in mathematics may be related to aesthetic as well as being a function of understanding a proof.

3.3 *Mathematical content: examples at the 'higher school level'*

In this part of the chapter, I study four typical items of content in higher school mathematics. These examples, each of which exemplifies an important part of the curriculum, are intended to illustrate the sort of concepts students are intended to understand, rather than point to exhaustive curricular categories. The areas of the curriculum which I have chosen to illustrate are the following:

- ◇ *axiomatisation*: through the example of the game of sprouts
- ◇ *modelling*: through the example of modelling projectile motion
- ◇ *infinite processes*: through the example of finding limits to sequences
- ◇ *symbolic manipulation*: through the example of manipulating secondary variables

These aspects of the curriculum are crucial to the 'twelve to twenty' curriculum of higher school mathematics.

In the first part of this chapter the focus was on the experience of mathematical work. In this part, the focus is on the substance of the mathematics which I teach. The final part relates an example of pedagogical content knowledge as it is experienced. In the first part, I presented calculations and reflections on those calculations from my own work, as that was the 'nearest to experience' I could get. Now, I analyse this 'mathematical substance' through activities which I have used as a teacher. Then I look at how this 'substance' can be mediated in the practice of teaching.

3.3.1 Axiomatisation

Mathematics is contextualised within many games (see, for example, Beasley, 1989) and the playing of games. One such simple one is 'Sprouts'. It is a well known game used in British schools which I have used with pupils on several occasions. After outlining how the game is played, I want to identify 'mathematical substance' players could potentially encounter and use.

The game is for two players; the one who makes the last legitimate move is the winner. The game takes place on a flat piece of paper: some nodes are drawn on the paper as spots. A node is 'closed' (can no longer be used) when three paths go from or to it and 'open' (still in play) when there are less than three paths from or to it. To play, the players take it in turn to join two 'open' nodes with a path and place another node on this path. No path may cross another.

There are (at least) two types of mathematical entity within this context: topological and axiomatical. Although the game can be played without naming them, the players are nevertheless working with these following mathematical entities:

3.3.1.1 topological equivalence.

This has two positive instantiations in this game: (a) in drawing a new path, the player uses the notion of 'the topological path between two nodes'; (b) in positioning of the new node on the just-drawn path, the player uses the notion of 'a-metrical betweenness'.

I note that this notion of topological equivalence (of paths and points on them) has a readily available perceptual representation in drawings. So, there are empirical checks on the legitimacy of moves. From the learner's point of view the concept of topological equivalence is grounded empirically. From the theorist's point of view, a distinction needs to be made between the empirical topological concept and the mathematical one (which can be defined in terms of homotopy). From the educationalist's point of view it is essential that a connection can be forged between them.

3.3.1.2 axiomatization: rule recognising and following.

Other mathematical objects encountered in playing sprouts are (a) the very rules that define the game; (b) the consequences of those rules.

Another way of saying (a) is that the game itself is a mathematical object. If this is granted, then could any game be considered a mathematical object? Not quite. 'Game' has social as well as mathematical meaning. Playing a game need not have tightly defined rules the way Sprouts does. Just as there needs to be a theoretical connection between the empirical notion of topological equivalence and the mathematical one, so there needs to be a theoretical connection between the social experience of playing a game and the notion of rule-boundness.

(b) includes two different aspects: the sequence of moves that will ensure a win; and the reasons why that sequence of moves ensures a win. For example, in a game starting with 1 node, the second player always wins. 'Proof' is by exhaustion! In a 2 node game, either the second player wins at her second move or the first player wins at his third move, depending on the play. Can either player force a win in a 2 node game? This is a question that can be put into propositional form and have associated truth value.

From the Sprouts context, I have pointed out two types of mathematical object: topological - 'equivalence' - and axiomatical - realised as 'rules-procedures-consequences' in games. The former have, for the inexperienced student, a legitimacy in empirical observation and the latter in social interaction.

I want, now, to consider another example that, in some sense, is an 'object' at the end of a chain of 'abstractions'.

3.3.2 *Modelling*

Like 'sprouts', mathematics is contextualised in projectile motion. In this case, the nature of a projectile is not defined by arbitrary rules, but it is descriptive and predictive of our physical environment. To understand basic projectiles requires physical as well as symbolic understanding.

Projectiles are understood bodily before we study mathematics. To point this out to students, I usually start doing projectiles by bringing in a ping pong ball projector (a. k. a. gun) to the class (the members of which are typically 16 to 17 year olds or mature students at that mathematical level). I then shoot ping pong balls at various people. If my memory serves me correctly, no student has ever failed to catch one of these balls, when one was projected towards them.

Because we 'know what happens' when a ball is thrown, it is relatively easy to motivate the students to reflect on how precise their understanding really is, and what are their underlying assumptions in modelling projectile motion. However, that does not imply any theoretical knowledge of the mechanical situation: ask them to draw a diagram of the situation, indicating the forces, and they frequently mark the net force on the ball in the direction of motion. This sort of response is not unusual! It was the most frequent response, 46%, given by Roper's sample ($n = 123$) of sixth form college students, (Roper 1985, pp 32-35), to whom a similar question was asked.

The mathematical content involved in analysing basic projectile motion boils down to vectors and derivatives together with the algebraic skills required to actively symbolise these entities. The physics content is Newton's Laws. What is involved in understanding

Chapter 3: Phenomena

projectiles is, initially, an understanding of the model, $\ddot{\underline{r}} = -g\hat{j}$ ¹³, where $\ddot{\underline{r}}$ is the acceleration vector, g , is a numerical value for acceleration due to gravity, and $\{\hat{i}, \hat{j}, \hat{k}\}$ is the standard orthonormal basis for Euclidean 3 dimensional space¹⁴.

A ball is projected. The angle at which it is projected can vary and influences the projectile's range and the time the ball is in the air. This is bodily knowledge involved in catching. Parameters must be incorporated into the model. How? The initial velocity is a vector; vectors are intrinsic to the mathematics of the action that is a projectile. This initial velocity, or speed-at-an-angle, changes; the rate of that change is constant: $-g\hat{j}$. The mathematical deduction of the unchanging horizontal component seems paradoxical until the mathematical concept of vector component and the mathematical concept of vector acceleration are married. The embodied knowledge is 'what ever goes up has to come down' is modelled, (at any rate, for ping pong balls), as $\ddot{\underline{r}} = -g\hat{j}$; the notation is conventional but the experience is real.

In the following example, on infinity, there is no similar physical experience to the embodiment which underpins the abstractions of the projectile model. In what sense can our finite existence 'experience' infinity?

3.3.3 *Infinite processes, infinite objects*

One way that infinity can be experienced is through the notion of process. Here the indefinite repeatability is experienced as a potential; recursion is part of our lived experience, even though we cannot perform a recursive procedure indefinitely. Spreadsheets are a fairly recent technological tool which enable someone to see the fruits

¹³Sometimes air resistance is worth incorporating into the model, but for ping pong balls in the classroom, empirical calculations indicate that good predictions as to, for example, the range of the projectile can be made without taking air resistance into account.

¹⁴This notation is used to indicate the generality of the situation, rather than to suggest that this would be students' first encounter with a symbolic representation of the model.

of many stages of a recursive procedure easily. The cognitive development in mathematical learning is to accept the notion of infinity as existing in and of itself.

One of the first objectifications of infinity that a student encounters is that of limit of a sequence of real numbers. This mathematical concept of 'limit' is often a difficult one to learn, as I shall exemplify further in 5.1. the difficulty arises partly because of the associations English speakers have with the word 'limit' and partly because of the intrinsic conceptual leap from the finite to the infinite. For example, we can describe an infinite sequence of numbers, by a recursive process, which may, or may not, end up somewhere finite: the sequence 1,1,2,3,5,8,13,... clearly both goes on forever and (its terms) get correspondingly bigger and bigger. The sequence $1/1, 1/2, 2/3, 3/5, 5/8, \dots$ must also go on forever, but its terms are all less than 1. 'Does it actually get anywhere?'

There is an extensive literature on the process-object 'duality' in mathematical learning (Tall, 1991, Sfard, 1991, 1994, for example). However, to understand the point of my giving this example as typical of higher school mathematics, I would refer the reader to the mathematical text like 'Infinite Processes' (Gardiner, 1991). In the introduction, 'What's wrong with calculus?', Gardiner exemplifies mathematically how perception and physically experienced intuition can fail (he uses the classic problem of 'mis-behaving' infinite series, also considered historically by Kitcher, 1984, Gray, 1992). A question which this thesis will work towards answering is what is the nature, philosophically, of the sort of mathematical object whose physical instantiation we cannot be sure of (such as limits of infinite processes). This is the subject matter of chapter 6.

3.3.4 Symbolic manipulation

Symbolic manipulation becomes important at this stage of mathematical development. As some of the mathematical data presented in 3.2 illustrates, there are occasions where genuinely new insight can appear through 'mere manipulation' of symbols. As Pimm remarks "Algebra is about form and transformation", (1995, p88). Pimm goes on to distinguish the "alternative emphases", when working algebraically, of the "generative" and "descriptive" aspects of the algebraic symbols (p 90). The generative aspect of

Chapter 3: Phenomena

working with the symbolism allows new information from mere manipulation. The descriptive aspect is the referencing to a mathematical object by which the user can get a reality check.

There are many examples of students' failure to connect meaning, manipulation and object. One of the challenges of higher school mathematics is the incorporation of a hierarchy of variables. The phenomenon of students' appreciation of different symbols in algebra has been studied by Liz Bills (Bills, 1997a, 1997b). Students may well be aware of the generative and descriptive aspects of algebra, but this does not imply that they are flexible enough to move efficiently between them. For example, Bills (1997b) reports on the way particular symbols, m in co-ordinate geometry, for example, have their meaning controlled by the role of that variable in ossified contexts: ' $y = mx + c$ ' in this case. This limits the possibilities which students are able to initiate: "[the novice's] ability to perform a task may depend crucially on its being expressed in terms of ... familiar notation." (p 80).

Liddie's problems with integration, which involved two levels of variables, further illustrate Bills's analysis and gives a specific example of a student not synthesising meaning, manipulation and object:

Liddie, a student on a two year PGCE conversion course, was aiming to use the technique of 'volume of revolution' to confirm the formula for the volume of a right circular cone of

height h and radius r . She had written $\int_0^h \pi y^2 dx = \int_0^h \pi \left(\frac{r}{h} x \right)^2 dx$, where she had set up the

cone as the line $y = \frac{r}{h}x$ rotated about the x axis between 0 and h . However, although she

was happy with $\int x^2 dx = \frac{x^3}{3} + c$, she could not deal with the right hand integral above.

Even substituting specific numbers substituted for r and h did not automatically help her. Her problem seemed to be that the expression combined brackets with fractions and squaring.

This exemplifies the way that mathematical entities are embedded hierarchically in a problem ($y = \frac{r}{h}x$ and ' $\int x^2 dx = \frac{x^3}{3} + c$ ' in $\int_0^h \pi \left(\frac{r}{h}x \right)^2 dx$). It also illustrates that solving the problem requires symbolic manipulation in different ways.

3.4 Pedagogical mathematical knowledge exemplified and analysed

My aim in presenting this final 'data set' is to help distinguish knowing how to *teach* elementary differential equations from knowledge of the content of differential equations. As I have already, in 3.2, exemplified working with mathematical ideas quite liberally, and, in 3.3, characterised key features of higher school mathematics content (3.3), I make this distinction by pointing to a kind of knowledge which is specific to mathematics teaching. I give an analysis of this mathematical 'pedagogical content knowledge' (PCK), (in the sense of Shulman, 1987), illustrated by means of a teaching episode. This kind of knowledge is central to teaching but not to mathematical practice.

The analysis, with the account of the teaching episode, was reported in an article on teacher's mathematical PCK (Rodd, 1995). In the present context, what I want to emphasise is the multifarious way mathematical knowledge is interpreted and represented for teaching. Part of such interpretation is planned and part is serendipitous or spontaneous.

3.4.1 Forward planning for differential equations

My first order planning for this lesson for 2 year PGCE students included the topic: an introduction to first order differential equations, and choice of resources: (most significantly) computer graph-plotters. This meant that the focus of the lesson was on first order equations which had a representation using the computer software, OMNIGRAPH, I had available. It also meant that I had to negotiate getting the computer room.

My second order planning started by deciding which specific differential equations the students should solve to start with, given that they had OMNIGRAPH available as a tool. The questions chosen for these students to work through were sequenced so that they each

Chapter 3: Phenomena

worked a different aspect of the mathematics that I wanted them to think about. I asked them to work in pairs so that opportunity for debate and discussion was easily available. I wanted the students to be fluent with the notation both in 'instrumental' and 'relational' terms (Skemp, 1976). To this end, I selected differential equations which I hoped would draw their attention to features of the software, the notation and the nature of the solutions.

The third order detail consists in the specific questions. The first equation they were asked to work on, $\frac{dy}{dx} = x$, was a familiar relation, but the Leibniz notation had not been used much before. The software displayed a representation of this relation as 'compass needles': short straight line segments of gradient $\frac{dy}{dx}$, positioned at enough (x, y) to give an impression, on the screen, of a varying 'field'. This was new to most of the class. A subsequent question was $\frac{dy}{dx} = y$ which was designed to draw their attention to non-polynomial solutions from notationally very simple differential equations. Another question was $\frac{dy}{dx} = -\frac{x}{y}$, which was included in order to start developing the technique of 'separation of variables'. Then I asked them to create and solve their own questions which may or may not be modelled on the various types mentioned above but would (probably) be limited by the I.T. that was being used.

From this description of my 'forward planning teaching decisions' I should like to make some observations:

- (i) I had to think about differential equations as a whole (as far as I could) as well as the detail of solving them.
- (ii) I encouraged the students to 'explore' and discuss solving differential equations, but I was reinforcing, particularly through the software, the received conventions and correct solutions.

- (iii) I offered scope for them to be as imaginative as possible in their creation of their own differential equations, but I expected them to work on solving these equations analytically if possible, using standard techniques and known results.

3.4.2 *Responding in the moment to students' mathematical work in this lesson*

As well as the forward planning, teachers respond to individual and group needs as they spontaneously arise. For example:

I had quite dissimilar conversations with two different students about the same question: Ahmad said that the 'compass needle' (on the OMNIGRAPH differential equations menu) was "a bit of the curve" from which we talked about how 'flat' was 'locally flat' and started working on local curvature. This level of sophistication was quite different from that of my conversation with Martin, who was overwhelmed by all the 'little lines'. Martin and I

looked again at the $\frac{dy}{dx} = x$ example represented on the screen and considered the question 'what $y = f(x)$ could satisfy $\frac{dy}{dx} = x$?' Whereupon he moved his finger on the screen in the shape of a parabolic solution curve, hazarded " x^2 ", checked by differentiating this guess and subsequently adapted his solution to the correct one.

From this description of my 'response in a moment', I should like to make some observations:

- (i) For the first student, I drew on my understanding of linear approximation in general, whereas with the second I homed in on explaining the software's representation of a tangent and how, specifically, to get a solution.
- (ii) I encouraged the students to 'explore' and discuss solving differential equations and related concepts, but I was reinforcing, particularly through the software, the received conventions and correct solutions.

Chapter 3: Phenomena

(iii) I offered scope for them to be as imaginative as possible in their questions and as focused as they wished when seeking help, but I expected them to work towards a fluency that included using standard techniques and known results.

In the observations for both the 'forward' and the 'momentary' teaching decisions I note that in

(i) I viewed the mathematical topic on occasion 'holistically' and at other times 'atomistically' (focusing in on detail)

(ii) I worked on the students making meaning for themselves through 'negotiation' in discussion and adapting their ideas when using the graph plotter and I checked that these meanings were developing to the 'precise' received mathematical concept.

(iii) I gave scope for 'creative' individual expression - whether in imaginative associations or in request for detail - but also wanted to facilitate 'mechanical' abilities (e.g. ability to employ algorithms where appropriate).

This is a brief analysis of mathematics-PCK. It illustrates how that part of teaching practice exemplified can be deconstructed into forms three kinds of knowledge. The conceptualisation of these kinds of knowledge give an inkling of the complexity of PCK and its relationship with mathematical knowledge and teaching practice.

This analysis is brief because (a) the question of the nature of pedagogical content knowledge in higher school mathematics is a topic worthy of a thesis itself; (b) because it is a digression from the central question of what it is to come to know mathematics. I have included this account because it indicates how the experience of doing mathematics (3.2) and the content of mathematics (3.3) can be brought together through employing another kind of knowledge: PCK. Obviously, that is what teaching is supposed to do!

3.5 Summary

The principle purpose of this chapter is to provoke and describe mathematical experience, then to prompt reflections on this experience which may stimulate philosophical

considerations about mathematics. Experience of doing and reflecting on mathematics is distinguished from awareness of the mathematical nature of the higher school mathematics curriculum. And it is distinguished from knowledge and experience of teaching mathematics. To what extent knowledge of teaching mathematics requires both background mathematical experience, along the lines expressed in this chapter, together with an awareness of the intrinsic mathematical content of the curriculum, is an interesting question which I may pursue at a later date! In this research my focus is on the philosophical issues arising from doing mathematics because they draw attention to the epistemological issue of how students are to access this experience and the ontological issue of what it is the students are working with when do enjoy mathematical experience.

4. Chapter 4: Philosophical Preliminaries

4.1 Introduction

“There is no there, there” attributed to Gertrude Stein concerning Oakland, California.

The purpose of this chapter is to locate the philosophical tradition relied upon in the subsequent three chapters. A 'philosophical tradition' is obviously multi-dimensional and no incidental analysis, like this, could give an adequate characterisation. In particular, it would take me far too far afield to attempt to do justice to an historical analysis of this established practice. This chapter will just highlight two aspects: the register and questions of interest. The register¹⁵ is the language, with the technicalities and nuances specific to this sort of philosophy, in which discussion takes place. But I do not present a self-conscious analysis of philosophy-talk here. Rather, I aim to use a philosophic register 'naturally' i.e. to communicate philosophical ideas. If the reader can stomach this philosophical excursion, the following three chapters should be straightforward reading!

The other dimension of the philosophical tradition described is the explicit content of this chapter: a sample of the background philosophical issues and questions which fuel philosophy of mathematics in the British-American analytic tradition. This sample is no random collection, but consists of issues and questions which contribute conceptually to the main thesis.

¹⁵The technical linguistic term 'register' indicates that a word or phrase can have a different nuance in different contexts: "Registers have to do with the social usage of particular words and expressions, ways of talking but also ways of meaning" (Pimm, 1987, p 108), indeed, "certain phrases and even characteristic modes of arguing that constitute a register" (ibid. p76).

4.2 Statement of overall thesis

In chapter 1, I set the scene for this project as a whole. Then I explained why the approach was to be philosophical in chapter 2. In chapter 3, the substance, or phenomena of the ground-level, of the enquiry was exemplified. Also in chapter 3 the passage from this ground-level of doing or teaching mathematics to the meta-level of philosophical reflection on the basic phenomena was traced. Now this chapter develops some philosophical preliminaries for the thesis proper, which is developed in chapters 5 to 7.

The thesis which I want to explain and defend has three parts, or sub-theses. These sub-theses are presented generally as I, II and III, and interpreted for mathematics in education as i, ii and iii:

I. There are distinctive mathematical warrants

II. Realist mathematical ontology is both defensible and compatible with educational interests

III. Some mathematical knowledge is non-propositional action knowledge

These assertions, I, II and III, are a foundation for the following propositions specific to mathematics in education:

- i. ways of reasoning at this mathematical level include deduction, quasi-empiricism and visualisation, and that students need not only to learn these processes, but also that these processes are the ones which serve to justify mathematical propositions
- ii. ontological commitment to the content of higher school mathematics is integral to a student's progress and a consequence of realism in mathematics
- iii. learning mathematics involves developing a capability to execute some mathematical procedures with 'automaticity'

4.2.1 *Some underlying issues*

Fundamental concepts which are used within the argument for the thesis include *truth*, *existence* and *intention*. These are vast conceptual domains. In practical terms, the content discussed in this current chapter must be focused on these concepts' application to the thesis in question.

The first thesis concerns warrants - claims for knowledge. When someone 'knows' that p , where p is a proposition, then this implies the truth of p . This is why a preliminary for a discussion of warrants is a discussion on the nature of truth. The role of mathematical warrants in learning mathematics is the subject matter of chapter 5.

The second thesis is about what exists. So a preliminary for this is the theory of what exists: ontology. Thus traditional ontological classifications are presented. A broad explanation of the lack of suitability of conceptualist and nominalist schools is given; realism, as applicable in mathematics education, is developed further in chapter 6.

The third thesis concerns knowledge-as-fluent-action. The difference between a laborious application of a mathematical routine and an expert fluent one can be characterised, in part, by the concept of intention. Intention is a concept from the philosophy of mind used to discriminate between instinctive-like actions and explicitly planned actions. The question of whether some mathematical action may be claimed as knowledge is discussed in chapter 7.

4.3 *Preliminaries for I*

The first part of this thesis discusses mathematical warrants and makes a claim that for mathematical knowledge particular forms of reasoning or justification are required. To a professional mathematician or philosopher this is a commonplace, but to a novice, typically a person in their early teens, the idea that different forms of reasoning are associated with different disciplines is by no means obvious. In chapter 5, I present ideas about what a justification consists of and why logical forms, and others, are of a particularly reliable type. Warranted belief is integral to knowledge. And the meaning of knowledge presupposes that what is known is true. So some discussion of truth is preliminary to a theory of warrant for learning mathematics.

At a first approximation, theories of truth can be classified as alethic or epistemic. The former classification is sometimes called ‘realist’ (Alston, 1996); a proposition is true in case it makes a statement about the way the world is. An epistemic theory of truth, on the other hand, decrees a proposition to be true if there is a way of explaining how this ‘truth’ can be known (ibid.). In the following sections these two conceptions of truth are discussed further.

4.3.1 Theories of truth relevant to mathematics in education

For the research mathematician, there may well be strong motivational reasons to adopt a realist conception of truth. For the aim in that enterprise is to find out whether a proposition is true or false: ‘does this solution converge?’ ‘does the solution converge ‘fast enough’?’ Clearly, when one moves to mathematics in education there are (at least) two interweaving discourses: that of mathematics and that of education and how propositions are warranted in these two disciplines is different. This is follows from the nature of the disciplines: one is an exact science one is a social science. It is arguable that in mathematics there are propositions the truth of which is not a matter influenced by humans, in education such is not possible. The aim here is to discuss mathematical truth from a mathematics learner’s perspective, in a way. This means that, if a realist conception of truth for mathematics is used, this is essentially a different concept (but not different meaning), from a conception of truth appropriate to education¹⁶.

In education, epistemic issues, how things are known, cannot be avoided. ‘Coming to know’ must involve beliefs, and whether certain doxastic attitudes are knowledge-like. This means that epistemic concepts like warrant, evidence, reliable, rational, etc. are important notions in this area of enquiry. Does this mean that a theory of truth pertinent for the discussion of ‘coming to know’ must be epistemic? An epistemic theory of truth is one for which truth is intimately entwined with how that truth is assessed; the knowability - the route to truth - is a function of processes performed by

¹⁶ I think that this problem which arises by trying to marry different conceptions of truth can be avoided by assuming conceptualism, (see 4.3.4). Perhaps a good reason for adopting conceptualism is to avoid

epistemic subjects. Before answering this question, I want to discuss, briefly, the relationship between ‘epistemological’ and ‘psychological’ in general and clarify the distinction between the epistemic and psychological subject in particular.

As a note of caution, the term ‘epistemic’ has different connotations dependent on whether it is used to denote a ‘subject’ or a theory of truth. An ‘epistemic subject’, (characterised below), is a ‘putative knower’, whose way to knowledge can be described in theoretical terms. An ‘epistemic theory of truth’ requires a mechanism for a putative knower to ascertain the truth value of a given proposition.

4.3.1.1 Epistemic or psychological

The science of psychology - the systematic study of the human mind - is a fairly recent science. There was not such a discipline in Ancient Greek times. Ancient Greek thinkers studied epistemology - the theory of knowledge - but not the discourse that tried to explain how knowledge develops, as cognitive psychology can offer today. Discussions of the nature of knowledge are plentiful in the writings of Plato, Aristotle et al. And so there was, in this tradition, a notion of the ‘epistemic subject’, (the slave boy in *The Meno*, for example), which Grayling equates with “the putative knower” (Grayling, 1996, p 40). The notion of the psychological subject - ‘the putative cognizer’ - is much more recent. Indeed, when it comes to mathematics learners, Fischbein (1990) argues that the psychology of mathematics education has only been recognised within the last thirty years as a sub-discipline of psychology.

Whereas the Greeks assumed existence of a pre-experiential knowledge - e.g. Plato’s forms - the seventeenth and eighteenth century British empiricists, principally Locke, Berkeley and Hume, based their epistemologies in subjects’ experience. The British empiricists’ thinking tended to merge the notions, raised above, of the epistemic and psychological subjects¹⁷. They did this by developing theoretical ideas about the mind and the mechanisms by which knowledge might be attained. Hume’s theory on these

this problem, but I think it is possible to give a coherent account while holding different conceptions of truth.

¹⁷ There may be other schools of thought which also bring these concepts together. For example, Appelbaum, (1996), suggests that Hume’s theory of mind in some ways mirrors a Vedic theory, (p13).

matters is both detailed and curiously modern. Firstly, Hume prioritised experience, and awareness of experience, in his enquiries; secondly he questioned the notion of a knowable reality; thirdly he challenged the whole notion of scientific, ‘natural’, necessity:

“the necessary connexion betwixt causes and effects is the foundation of our inference from one to the other. The foundation of our inference is the transition arising from accustom’d union. These are, therefore, the same” (Hume, 1739/1978, p 165).

Hume’s scepticism about knowledge resulted from the strict logic of his conception: ironically, in his aim to privilege experience of reality, his reason concluded that no ‘direct representation of reality’ could be known!

The meaning of the term ‘epistemology’ has a particular nuance in the field of mathematics education which, I think, is subtly different from the meaning used by the philosophers, (like those cited above), with whose writings I am trying to work. The difference in ‘philosopher-epistemology’ and ‘mathematics education-epistemology’ is on two levels, epistemological theory and ontological assumptions. Firstly, most philosophical discussions on ‘coming to know mathematics’, (the epistemological theory level), focus on rationales for how a stimulus can be warranted and therefore be knowledge-like. This is exemplified in Plato, (e.g. the Meno), in Wittgenstein’s language based theories (see 7.2.5) and in Giaquinto’s visualisation (see 5.4). In mathematics education this level of discussion is usually scientific, with theories being developed from experimental data, as any perusal of the proceedings of PME will evidence. ‘Epistemology’ in this sense is a branch of psychology. Secondly, the other, ontological, level is considered a separate (though related) issue by philosophers, (see Grayling 1996), but part of the epistemological debate by mathematics educationalists (see Vergnaud, 1990). By drawing the ontological and epistemological issues closer together, I think that mathematical educationalists have lost some of the subtleties to do with the nature of acquiring mathematical knowledge. The distinction between ontological and epistemological issues may be able to be re-

marked by assuming a realist ontology with an appropriate, but distinct, psychologistic¹⁸ epistemology.

So what is the difference between the epistemic subject and the psychological subject?¹⁹ A given individual can be considered as an epistemic subject or a psychological subject. At a basic level, knowledge, or information, about the psychological subject is obtained through scientific investigation; it concerns behaviours, attitudes and understandings. Epistemology addresses the issue of ‘knowledgeableness’ either of specific individuals, like students, or of an abstract human subject, or indeed of ‘communities’; these are epistemic subjects. Psychology deals with (among other things) cognitive development through the functioning of an individual’s brain-body-social group, whereas epistemology deals with knowledge, routes to knowledge and pre-knowledge mental (or, perhaps, bodily) states including experience and reasoning.

4.3.2 *Theories of truth*

In natural science investigation, ‘truth, the whole truth’ is formally unobtainable: scientific progress includes revision of previously accepted theories, no observation can be recorded with complete faithful accuracy and not all information of the world could be processed. So if science does not yield such truth, is ‘truth’ just a social construction? Some authors do indeed assert this relativist position. Rorty (1980, 1990) and Bloor (1976) are examples from philosophical circles; Ernest (1991, 1997) has explored a social constructivist metaphor in the mathematics education context. However, for the thesis which I want to defend some sense of ‘realist truth’ is implicit. I shall now present some conceptions of ‘realist truth’ together with some alternatives.

¹⁸ ‘Psychologistic’ is the term Kitcher (1984) uses to connote a cognitive dimension to a theory of knowledge.

¹⁹ This question was raised by a participant at my session at PME 1997.

Referential realism

However, rejection of the possibility of finding and knowing one has found absolute truth in science does not force one into a relativist position. Harré, (1986), develops a ‘scientific realist’ theory which eschews relativism. In his theory, he rejects the ‘bivalence’ categories of truth and falsity because of the practical impossibility of doing a total check. Instead, Harré develops a theory of reference in which “reference is established by achieving a *physical* tie between embodied scientist and the being²⁰ in question” (emphasis in original, *ibid.* p68). Harré has explained that his motivation for taking a scientific realists’ tack to counter the logical positivist tendencies in the middle of the century (Harré, 1960) was that he had been a practising scientist (an applied mathematician). Those theorists who took an alternative, sociological, tack to counter the positivists, like the positivists themselves, did not generally have a scientific practice as a touchstone for their theory. The experience of the *material* practice of science helps in the conceptualisation of a scientific reality. And while a purely social practice does not require material referents, a material practice (like science) does. This ties in with my emphasis on mathematical experience being crucial for being able to discuss the nature of mathematics in the developing practice of learners. The ‘physicalist’ realism, which I adopt as being able to support a suitable ontology for learning mathematics, (see chapter 6), is ‘indispensably’ tied to the material practices of sorting, counting and predicting.

Harré’s referential realism may be suitable to explain scientific knowledge, i.e., as an epistemology for natural science, but does not the notion of ‘truth’ have a privileged role in mathematics where we can say ‘for sure’ that a proposition is true or not? I think that it does. And, indeed, from the ‘evident truth’ of elementary mathematical propositions, the very meaning of ‘true’ may evolve for some English speakers. This is because mathematics is both scientific and semantic: there are material and causal touchstones in mathematics *and* there are logical forms. This idea is very similar to

²⁰ ‘being’ in this quotation refers to a ‘being’ of one of three types: from common perception; potentially observable given technical tools; beyond all human observational capacity, (see, Harré, 1986, p59).

the one espoused by Zheng (1992, see 6.7). The relationship between the material and the formal is more complicated than a ‘weft and warp’ image could connote, for, as I shall argue in 5.4.3, some basic logical forms also have a material basis. Inasmuch as mathematics is a science, the usual caveats apply with respect to truth claims. Inasmuch as mathematics includes formal systems, the truth or falsity of a given proposition is systemic. The problem comes when the ‘truth’-concept of formal systems is attributed to fallible scientific knowledge. The logicians in the first half of the twentieth century showed that formal systems’ interpretation was not a straightforward ‘intuitive’ business.

4.3.2.2 *Epistemic theories of truth*

Harré’s conception of truth is epistemic. This is because he defines truth as ‘moral’: we take something to be ‘knowledge’ (so true) if it is commended by those we trust. Thus the mechanism for a truth claim is specified. To be able to act on such a commendation, requires a common discourse as well as a common standard of sincere reporting. Harré rejects naïve ideas of truth and falsity in science, as an alethic theory seems to require, for the straightforward reason that they are impractical. Instead he advocates that “the qualification by name is a kind of ‘epistemic equivalent’ of assessments of truth and falsity...The moral status of persons determines the epistemic status of their results” (op. cit. p85-6). Harré locates the psychological development of knowledge within educational practices of the relevant community; the epistemic result of a ‘good education’ is that the student will ‘take-as-true’ teachings from those with good reputation. In this way Humean scepticism is avoided, but at the loss of the possibility of the knowledge-making potential of individual experience and the realist conception of truth.

Epistemic theories of truth are the contrary of the correspondence-type theories sketched below. In an epistemic theory, whether a proposition is true is related to how a ‘truth’ is known from the question of how truth is to be known (or at any rate believed). ‘Coherence’ theories, in which “it is said that the mark of falsehood is

²¹ At a seminar at Linacre College, Oxford, 6/12/97.

failure to cohere in the body of our beliefs” (Russell, 1912, p70), are generally epistemic with regard to truth. There are two types of these theories: the early-century version, where the ‘body of beliefs’ is construed as a giant system, underpinned by logical relations, and the late-century version. In the latter the metaphor of corporeal ‘body’ has given way to that of ephemeral ‘discourse’, which is underpinned by, (and dialectically underpins), meanings shared by discourses. In either case, though different, the road to truth lies within the ‘body-system’ or the ‘discourse-meaning’.

Turning now to the ‘body-system’, one of the most pervasive notions concerning truth and realism is that the truth value of statements can be determined by a ‘correspondence’ between meaning, of the statements, and relations between ‘objects’, in ‘objective reality’:

4.3.2.3 Correspondence theory

Nearly all realist philosophers crave some sense of correspondence in their theorising! Even the philosopher Michael Devitt, who argues that “realism does not strictly entail any doctrine of truth at all”, (Devitt, 1984, p35-40), goes on to describe how a version of ‘correspondence truth’ could work, (pp96-9). In this section, I outline forms of ‘correspondence theories’ starting with a very formal definition and ending with the ‘minimal realism’ favoured by some contemporary philosophers.

To begin with, a description of this theory is given by the philosopher A.R. White in abstract terms as follows:

“By interpreting the correspondence between the statement that *p* and the fact that *p* as a correspondence of what is said *to* what is a fact, that is, as a mere one-to-one correlation between these items - without any hint that one resembles or fits or is structured like the other - the Correspondence Theory remains faithful to the basic and indisputable principle that *p* is true only if *p*.” (White, 1970, p108, emphasis in original)

A formal description such as this avoids the nitty-gritty epistemological issues of access to the very fact of the so-called correspondence, or even, that there is an issue about how we ‘come to know’. In the statement above ‘*p*’ can be any sort of statement

- the examples White tends to give are historical 'facts' - this level of generality makes it difficult to discern the nature of the correspondence, and does not 'sell' the theory at all.

Russell (1912) advocates a correspondence theory of truth and does acknowledge the problem in specifying the very correspondence on which the theory relies: "truth consists in some sort of correspondence between belief and fact. It is, however, by no means an easy matter to discover a form of correspondence to which there are no irrefutable objections." (p 70). Russell then goes on to define this 'correspondence' in terms of something he calls a 'complex unity' (p 74). However, this new construct does not seem to have either further explanatory powers or describe a causal or logical mechanism of correspondence.

Despite the inadequacies of these 'correspondence-notions', roughly speaking, the correspondence theory of truth captures a common-sense idea that there is a direct correspondence between propositions and facts about things. For, example: 'three points determine a circle'²² is *true* only if three points do determine a circle. The correspondence theory is, perhaps, a first approximation to a viable realist theory of truth, but, as I have indicated, a central difficulty is how this so-called 'correspondence' is known. The theory is a sketch - 'we can talk about things' - but does not bear up under close scrutiny of the detail of how information is tested for veracity. As Wm. Alston says: "a robust correspondence theory must develop an explicit account of propositions ... and facts so as to be in a position to spell out what correspondence amounts to." (Alston 1996, p33). Alston rejects 'traditional' correspondence theories of truth because of the strong requirements placed on the mechanism of correspondence. Instead he develops a 'minimalist' account of truth within his theory of alethic realism (ibid.), which does contain a weak notion of correspondence.

²² Either allow the limit case, a line, when the points are co-linear; or require the three points to be non-co-linear.

4.3.2.4 *Realist theories of truth*

Correspondence theories of truth which make strong claims in terms of the connection between fact and belief are difficult to justify. This is why contemporary philosophers have moved away from such theories. Contemporary philosophers tend to favour either coherence theories - in which 'truth' is a function of what is agreed - or weaker realist theories.

One such 'weaker theory' is the 'disquotational theory of truth' which is closely related to Alston's 'minimal realism'. This is also called a 'deflationary' theory of truth. This theory of truth is designed to avoid objections to the 'traditional' correspondence theories by 'deflating' the whole issue; it is a linguistic device which declares p true if and only if the conjunction of all the underlying requirements for p are true. Maddy, whose writings I discuss at some length, finds it acceptable for her purposes. She gives two examples to illustrate the theory: let p be 'everything in the Judeo-Christian Bible', then p is 'disquotationally' true if the conjunction of all the sentences in the Bible were true. Her other example concerns the knowability of arithmetic propositions, and in this case requires an infinite conjunction of underlying requirements. Maddy's point is to show that a 'full blown correspondence theory' is not required for her realism, (Maddy 1990, pp 17-19).

Davidson, in conversation with Papineau, (Davidson, 1997P), describes such a theory of truth as 'thin', for it can only apply to one's own sentences and, importantly here, does not apply to beliefs. Davidson goes on to say that "truth just is"! Truth is as basic a concept as you can get and one which cannot be defined. Thus he seems to avoid the pitfalls of either realist or coherentist camp. Davidson's position on truth is that it is a "property of certain sentences attributed by an empirical strategy and interpretational theory" (ibid.) which suggests that Davidson holds with some aspect of scientific realism which is sensitive to linguistic constraints.

Alston takes pains to distinguish the *property* of truth from the *concept* of truth (e.g., p37, p 41) because it is the "ordinary concept" of truth with which he wants to work which may have features we may not be able to find out about:

“A property may have features ...that is not reflected in our concept that picks out that property. ...The essential nature of water is to be H₂O, although our ordinary concept of water is in terms of its observable properties. ... in the same way the *property* of truth may have various features that are not reflected in our *concept* of truth ... In particular, it may have the features embedded in the correspondence theory, features on which the aspect embodied in our concept supervenes.” (p37/8)

Alston considers that his minimalist theory is quite close to ‘deflationary’ theory. In deflationary theory, the “truth talk” that can accompany discussion of a proposition, is be ‘deflated’ and the property of truth-attribution diffused by explaining the property attribution in another way. This process is illustrated by Maddy’s examples cited above on the meaning of the disquotational truth of the Bible and of arithmetic propositions.

4.3.3 Summary

This section (4.2) has been a contained exposition of the notion of truth. I have presented truth as either epistemic or alethic²³ and shown that the boundary between these two categories is fuzzy. For example, Harré’s epistemic conception is bound to material scientific practice and Davidson’s direct realism requires some shared meanings for interpretation.

As another preliminary for the epistemological thesis, I, which is put forward in chapter 5, some subtleties in English words with the root ‘episteme’ are discussed. In particular, an ‘epistemic subject’ is a being who is potentially able to know something, knowledge requires a theory of truth, *but* the epistemic subject’s knowledge does not have to be bound by an epistemic theory of truth²⁴.

²³ Rorty (1990) anti-representationalism probably does not succumb to such a categorisation. Rorty’s view is hard to incorporate into a theory such as this in which I am trying to make distinctions between notions of truth and existence, for he explicitly denies the anti-realist/realist distinction which I discuss. Nevertheless, others (e.g. Putnam, 1990) attribute to him an anti-realist stance.

²⁴ This point will be quite important in appreciating Goldman’s cognitive epistemology which is discussed in chapter 5.

4.4 Preliminaries for II

The second part of this thesis discusses mathematical existence. In II a claim is made that mathematical entities do exist and that a tie to reality is through human beings' particular physical make up within the physical world. To a professional mathematician this may at first sight seem an abomination: mathematical entities are not physical! And to a novice, it is hard to imagine 'things' which are not touched, seen or otherwise sensed. In chapter 6, I present ideas about how mathematical 'abstractions' may indeed exist and that their existence presupposes a material reality which is the universe of which we are a part.

As I have related, the works I draw on are primarily in the analytic tradition of English speaking British, North American and Australian philosophers, which looks to Plato and Aristotle as 'founding fathers'. Indeed, discussion of the nature of mathematics has been going on since their era. Tiles (1996) distinguishes the basic positions of Plato and Aristotle as 'realism' and 'conceptualism', respectively: Realists, after Plato, posit the existence of mind-independent mathematical entities, whose existence is denied by conceptualists. Conceptualists, after Aristotle, construe mathematics as a product of the human mind's "innate relation-imposing capacities" (Tiles, op. cit. p332). There is also the 'nominalist' position, which denies the existence of any abstract entities - only space-time particulars exist. Nominalists employ mathematics as a formal structure but deny its intrinsic meaning. These three categories of *realism*, *conceptualism* and *nominalism* constitute the domain of discourse in analytical philosophy. (As I have said, I avoid other domains like mysticism or hermeneutics.) They are described by Quine:

realism, *conceptualism* and *nominalism* are "the three main mediæval points of view regarding universals...[they] reappear in twentieth century surveys of the philosophy of mathematics under the new names *logicism*, *intuitionism*, and, *formalism* ... *Realism* ... is the doctrine that ... abstract entities have being independently of the mind; *conceptualism* holds that there are universals but they are mind-made, [and] *nominalists*, object to admitting abstract entities at all, even in the restrained sense of mind-made entities." (Quine, 1953/64 p 192-3)

Although this categorisation may give an idea of what mathematical ontologies are possible it does not give an insight into what mathematics is. More specifically, the grand labels 'realist', 'conceptualist' or 'nominalist' do not, on their own, help us understand what a 'mathematical object looks like'. Nor do these categories shed light on how mathematics may be 'grasped', 'made meaningful' or 'worked with', respectively, which is the essence of the educational question. However, these categories can serve as a conceptual framework for further discussion for the doctrines of these schools have been, and continue to be, adapted throughout the ages.

Current thinking sees the boundaries of these categories, realism, conceptualism and nominalism, as fuzzy and this generates philosophical questions and new philosophical theories. For example, there is a current vigorous debate 'on the boundary' between the realism of Maddy and the nominalism of Field (see Maddy 1989). Kitcher could be interpreted as borderline constructivist, i.e. a conceptualist, given the above categories, (see Kitcher 1984) as well as some sort of realist. And the arch-intuitionist (and so conceptualist) Dummett accepts a large measure of logicism (see Dummett, 1992). Dummett's logicism, however, is more 'nominalistic' than 'realistic'. The point is, the mediæval positions still serve as a foundation for debate about the nature of mathematics and their meanings are still evolving under interpretation and the arguments of contemporary philosophers. In mathematics educational literature, however, the positions of nominalism and realism have been over-shadowed by the popularity of versions of conceptualism (see Sierpinska and Lerman, 1996, for a review).

As a 'philosophical preliminary', then, it is appropriate to outline the basic positions: realism, conceptualism and nominalism.

4.4.1 Realism and anti-realism

To set the scene, the position which asserts the existence of mathematical objects is contrasted with that which denies their existence: these positions can be described by the mutually exclusive terms 'realism' and 'anti-realism'. 'Anti-realism' was coined by Dummett, (e.g., 1992), and is similar to the term 'idealist' in philosophical usage. 'Realism' is also a technical philosophical term. Like any term that has a wide usage, it

is associated with a rich set of meanings for which no definition could capture its range of nuance. Connotations of the ordinary language notion of 'realistic' are to be eschewed; philosophers, such as Dummett, are fervent anti-realists. It is not necessarily un-realistic to be anti-realist! Young (1996), a "card-carrying anti-realist" does insist that anti-realism leads to relativism "of some form". By this account, one can realistically be an anti-realist, but not absolutely!

Dummett makes the realist/anti-realist distinction in mathematics to be about sentences and the possibility of ascertaining their truth values. This means both that the debate on mathematical entities has been put into a linguistic realm: the world is 'a collection of facts not things' and that the version of truth employed is epistemic. Indeed, the term 'verificationism' is used (ibid.) to describe Dummett's theory. In the following quotation, Dummett pinpoints his distinction between realism ('the way things are') and anti-realism ('the truth about our assertions'):

"It appear[s] to me evident ... that, interesting as the questions about the nature of mathematical objects, and the ground of their existence, may be, the significant difference lies between those who consider all mathematical statements whose meaning is determinate to possess a definite truth value independently of our capacity to discover it, and those who think that their truth or falsity consists in our ability to recognise it." (1992, p 465)

In his valedictory lecture (1992), Dummett explains that his coining the term 'anti-realist' was to aim to stimulate a research programme to investigate the structure of realist vs. anti-realist theories and to investigate similarities and differences between realist and anti-realist debates in different disciplines (p 463). For example, behaviourism, instrumentalism and phenomenism are anti-realist theories about the mind, science and the physical world respectively. Dummett argues that none of these defeat their realist counterpart. However, in the case of mathematics, his constructivist anti-realism is a serious challenge to realism of both the pre-theoretic common-sense kind as well as the philosophical theorists' conceptions. This is because, in his words:

"Mathematics was the most propitious field for the development of an anti-realist theory of meaning precisely because the gap between the subjective and the objective is there at its narrowest" (p 471)

I interpret this as meaning that a rational person's individual (therefore 'subjective') logic corresponds to formal (therefore 'objective') deduction. Hence Dummett is a conceptualist, and any argument for realism against conceptualism must address his semantic view.

4.4.1.1 *Realism*

There are many forms of realism, from Plato's to Putnam's (e.g. Putnam, 1990), but a collective feature of realist theories of the world is that "existence is prior to theory" (Harré, 1986, p5). In terms of contemporaries, Harré's 'modest realism' recognises that securing a scientific belief is, in some sense, a social activity. Nevertheless, Harré insists that "for there to be public reliability something must exist independently of whomsoever first found it." (p12). Putnam's 'internal realism'²⁵ eschews conceptual relativism which he considers the standard anti-realist position²⁶. Popper (1972) was also a realist. His metaphor of the 'mountain beneath the clouds' suggests a 'reality' which 'we seek'.

In a nutshell: to assert 'realism' involves asserting the existence of an external world and it is that external world which, in theory, is the ultimate arbiter of truth values.

Attraction to this philosophical realism can come from various sources. For example, Maddy indicates two distinct reasons for developing a philosophically realist thesis of the nature of mathematics: the first is because she accepts Quine's 'naturalised epistemological position' which asserts that science may be considered 'our best knowledge of the world', and so science is used as an epistemological foundation, (see 6.4.1.1 below); the second is that the phenomenon of mathematical practice includes

²⁵ This position, briefly, denies the 'God's eye view' of nature: Putnam claims to be a 'small *r*' realist (Putnam, 1990).

²⁶ "Rorty's view is just solipsism with a 'we' instead of an I" (Putnam, 1990, p ix)

“pre-theoretic realism” (Maddy, 1990, pp 1-5). This ‘pre-theoretic realism’ is well known. It is essentially the phenomenon of which Sfard speaks :

“[The] ‘natural’ state [of a mathematician’s mind is] the state of a Platonic belief in the independent existence of mathematical objects, the nature and properties of which are not a matter of human decision.” (1994 p 51)

These two premises - that science is to be an epistemological foundation and that mathematicians seem to work with mathematical entities - motivated this contemporary philosopher to develop her theory of mathematical realism (ibid.). A philosopher's job is to take the data that includes the reports of such mathematical experience and to make a philosophical account. A job for a mathematics educationalist is to scrutinise and interpret such a theory for its potential to give insight for the learning of mathematics.

The initial conception of realism can be refined further. Shapiro (1993) distinguishes between the question of mathematical existence and that of the properties of these purported objects as they are asserted in sentences:

"realism in two senses[:] First, it is held that mathematical objects, sets, exist independently of the mathematician. This may be called 'realism in ontology'. Second, the assertions of set theory have objective truth values, independently of the conventions, languages, and minds of the mathematicians; and the bulk of the assertions of competent theorists are true. Call this 'realism in truth value'." (p 455)

In the academic tradition of philosophy of mathematics (to be described in more detail below) there is a recent renaissance of realism in the form of 'physicalism' - that there is a scientific germ to mathematics (e.g. Irvine 1990, Milne 1994). This view is, in turn, in tension - but not in contradiction - with a Kuhnian conception of science as a function of a community, as seems to be advocated by the 'naturalised epistemologists' (of whom Quine is a founder member, see his 1969, pp 91 -113).

Details and criticisms of realist theories due to Maddy, (op. cit.), Kitcher (1984), and more briefly Resnik, (1993), and Bigelow, (1988), are given in chapter 6 as part of thesis II. In this preliminary chapter, a sense of anti-realism in the distinct forms of

conceptualism and nominalism is given through brief expositions of the theories of school-leading advocates of these respective positions. These outlines are of general theories of the nature of mathematics and application to education is not systematically attempted here. Specific problems inherent within conceptualism and nominalism when these theories are applied to mathematics in education are discussed within chapter 6.

4.4.1.2 Conceptualism: Dummett's semantic logicism

Conceptualism in mathematics is the view that mathematics is not in any sense part of the world around us. Instead, mathematics informs our conceptualisation of that world. This it does by being a descriptive and explanatory language. Indeed, Dummett's approach is to turn the ontological debate from "the disputed class of objects" to the "disputed class of statements" (Dummett, op. cit., p 465). In other words, he proceeds on the basis that anti-realist ontology can be subsumed under anti-realist truth-theory. This makes his theory essentially semantic and paradigmatically conceptualist, which is why I focus on his ideas through which to present aspects of conceptualism.

4.4.1.2.1 Verification of linguistic items is through language

The case that Dummett makes for focusing on language is partly based on his claim for what it is that can be tested:

"I recommended starting, not with the metaphysical status of the entities, but with the account to be given of the meaning of the statements. ... Since no means offered itself for deciding which picture of reality was correct, the more fruitful approach lay in determining which picture of meaning was, since in this case there was a theory of meaning to be constructed and a linguistic practice against which to test it." (p 465)

The formulation in these terms seems to me essentially dualist inasmuch as he seems to set up untouchable metaphysical entities, (a reality behind the linguistic picture), despite declaring that they are to be avoided. I do not think he can legitimately subsume ontology to sentential truth-value by fiat. Furthermore Dummett claims that:

"the significant difference lies between those who consider all mathematical statements whose meaning is determinant to possess a definite truth-value independently of our capacity to discover it, and those who think that their truth or falsity consists in our ability to recognise it" (p 465)

In other words, looking to 'linguistic practice' as the ultimate judge (rather than as Quine might advocate, 'science') is surely to pull a veil between the 'objects' of the cognising subject and the 'objects' of the world of which that cognising being is a part? If you believe that all that can be analysed is the froth of language, then the sea of existence will be ever inaccessible.

4.4.1.2.2 *Logic as paradigmatic reasoning*

Dummett agrees with the 'logician' thesis that mathematics consists "in the systematic construction of complex deductive arguments" (p 432). This was the theory initiated by Frege and developed by Russell and Whitehead and, as is well known, failed because of the set theoretic paradoxes, (a set defined such that it both is and is not a member of itself, for example). Dummett argues that these paradoxes are only devastating when one insists on the existence of abstract objects and that Frege's insistence on doing this was his down-fall. And although Russell and Whitehead "tried to construct foundations for mathematics in accordance with the more natural conception of logic as independent of the existence of any particular objects" (p 433) their approach required assumptions, like the axiom of infinity, which "could not be rated as logical" (p 433). Nevertheless, Dummett is attracted to develop a semantical logicist thesis, which both avoids Frege's referential problem of mathematical objects and Russell and Whitehead's extra axioms, because of the way logicism captures (some of) the feel of pure mathematical reasoning.

Dummett's attraction to the logicist thesis is, in particular, due to its explaining why mathematics "involves no observation" (p 432); demands stringent standards of proof; is so widely applicable and there is a sense of necessity about its results. If mathematics is logic and logic is formalised legitimate reasoning, then proof standards, applicability and feeling of necessity do follow from this conception. Dummett's position is nicely summed up in the following quote:

"The aim of representing a mathematical theory as a branch of logic is in tension with recognising it as a theory concerning objects of any kind, as its normal formulation presents it as being: for we ordinarily think of logic as comprising a set of principles independent of what objects the universe may happen to contain."
(p 434)

The dualism is apparent here again: for how can we imagine principles of reason outside the universe of which we (reasoning beings) are a physical part?

Dummett classifies abstract objects as contingent or pure. The former class depend on concrete objects in the world, the latter do not depend for their existence on empirical reality (p 437). For a contingent abstract object, he uses the example of the equator and does not exemplify (in this essay) a pure abstract object, perhaps 'God' would serve as an example, I am not sure. Anyway, Dummett then claims that "the significant distinction is not between abstract objects and concrete objects, but between mathematical objects and all others" (p 438) and this is because, he claims "the existence of mathematical objects is assumed to be independent of what concrete objects the world contains." (p 438). In other words, he aims to further narrow the "gap between the subjective and objective" (as quoted above) in mathematics by decreeing the content of mathematics linguistic. Mathematics is applicable because it *is* semantically coded rationality, not because numbers, geometry, chance, etc. exist, in some sense, in the physical world.

Dummett's view relies on the idea of (non-material) 'reasoning' existing without requiring the existence of any 'reasoner'. This is not the same as the view that the structure of forms of reasoning can be classified and, indeed posited, independently of any particular reasoner or group of reasoners (which do not need to be human). Dummett's claim seems to require the existence of disembodied reasoning - logic in the ether - which he then positions within human language. This seems too good to be true: this underlying ultimate reasoning - logic - is outside objects of the world but inside the linguistic constructions of (some) objects of the world. Human language's abstractions are contingent; the 'equator' is a good example, although relations between them can be necessary consequences of their meaning, for example the 'equator is circular' follows from its meaning. So the reasoning structure is of no different type. In

other words, I do not think he has driven a wedge between "mathematical objects and all others", precisely because of his reliance on language and the importance of reasoning - and so logical form - in language.

4.4.1.2.3 Dummett's conceptualism: summary and contextualised for mathematics learning

The two basic tenets of Dummett's which seem to underpin his conceptualist theory are (a) that we test meanings, and so truth values, through language; and (b) that mathematics is semantically coded rationality. In the spirit of this view, it would be consistent for mathematics learning to be restricted to logic and semantic formulations. Observations of scientific, 'real world', phenomena which present patterns for abstraction, would not be a suitable for mathematics instruction. But, as is well known, investigation of real world phenomena, either with 'manipulables', as modelling or through creative art work, are particularly suitable methods for learning mathematics (see below, 6.2.3.1).

4.4.1.3 Nominalism

Nominalism - the thesis that there are no abstract entities - is subject to many technical difficulties in attempting to reconstruct the 'mathematics of science'. But, perhaps the most obvious objection to nominalism, per se, is the charge of 'double-think': denying what you use. Denying mathematics when philosophising is one thing, but denying mathematics when involved in the process of scientific work is quite another. How can you deny what you so rely on? This is the 'indispensability thesis' attributed to the work of Quine and Putnam in the third quarter of this century, on which Maddy relies (see 6.4.1.1). Field believes that he can answer this 'double-think' charge with his logical reconstruction. I do not think one can answer the charge of 'double-think' if issues of epistemology, not just ontology, are considered (see 6.2.1). I shall trace a very brief, non-technical, outline of Field's carefully constructed theory, (Field, 1980).

4.4.1.3.1 Field's nominalism

Field's particular brand of nominalism is discussed because his case against the existence of mathematical entities is quite persuasive; I am not persuaded because the

concepts involved in mathematics are not epistemologically 'formal fictions', even if a case for excising them ontologically can be made. (This is explained further in 6.2.1.) Field's thesis is attractive because he grasps the nettle of mathematics' 'unreasonable effectiveness' and works to explain mathematical effectiveness even while denying mathematical existence! He is a scientific realist but not a mathematical one. Field's theory was motivated in part by a desire to eliminate "certain sorts of 'arbitrariness' or 'conventional choice' from our ultimate formulation of [scientific] theories" (Field 1980 p ix). To this end he developed his nominalist theory: "Nominalism is the doctrine that there are no abstract entities" (op. cit. p 1). Field's approach to show that mathematics, per se, does not exist is to show that the 'abstract entities' of mathematics are just not needed in physical theory. He co-opts Berkeley's term "fictionalist", and uses it to express his denial that the part of mathematics concerned with numbers, sets, functions and other mathematical objects, is true.

As mentioned above, Field is a scientific realist. He just wants to show that mathematics is not part of science. Truth values are applicable to propositions of science but not to propositions of mathematics. He distinguishes himself from other mathematics-denying positions like "doctrines which interpret mathematical statements about linguistic entities or about mental constructions. [For] such nominalistic entities do nothing toward illuminating the way which mathematics is applied to the physical world", (op. cit. p 6).

Field claims that the evident utility of mathematical entities is not evidence for their truth. It is possible to use mathematics as a convenient tool but, unlike the theoretical entities in science, no new claims can be made about observables. The crux of his argument for the denial of mathematical objects rests on the 'conservatism' of mathematics - all conclusions arrived at using mathematical entities "are already derivable in a more long-winded fashion from the premises, without recourse to the mathematical entities" (op. cit. p 11).

The notion of 'conservatism' is a familiar term in philosophy of science (op. cit. pp 16 - 19). Loosely, it means that information is neither added nor extracted with the incorporation of mathematical symbolism and processes, nor is any inconsistency introduced. Newton's laws together with Calculus are just Newton's laws. Where

Calculus, conceived nominalistically, should add nothing to the predicative import of the laws, it would just make calculations more manageable. Indeed, Kitcher (1984, p 231) reports that Newton's work itself was much more concise with the calculus. Of course this calculus was teeming with ontological assumptions, as well as inconsistencies, of the sort of which Field would like to rid us! Field claims "good mathematics is conservative" (op. cit. p 13): it is a logical system which, through symbolic succinctness, curtails reasoning so that scientific results are more easily obtainable.

4.4.1.3.2 Summary: Field's nominalism and crossing an epistemological 'gap'

One of the purposes of nominalism is to make ontological reductions: mathematical entities can be dissolved away using the nominalist programme. Yet because it would be impossible to reformulate all of science, there is always the possibility of a psychological reliance on the yet-to-be-nominalised parts of science. The number of entities to which one had ontological commitment may have reduced but there still remains a notion that there are true statements about abstractions (numbers, metric spaces, groups etc.). Field does not deny that some people have some mathematical conceptions; his point is that these so-called mathematical conceptions are either scientific or fictional notations. 'There is no mathematics in mathematics', as Gertrude might have said to Alice. From this point of view it seems as though we still need to learn mathematics 'as if it exists' and then later liberate ourselves from a formally unnecessary ontology. This is not satisfactory because of the unnatural requirement that we should deny what we understand and use.

4.5 Preliminaries for III

The third part of this thesis consists in an argument for a proposition which states that a certain kind of action constitutes an aspect of mathematical knowledge. This claim is more tentative than those made in the first two parts. In the first two parts, the concepts of mathematical warrants and mathematical objects are familiar ones in the philosophy of mathematics, whatever position is taken on them, and their application to mathematics in education is the novel aspect of the thesis. However, the proposition that action can be knowledge is not part of the standard domain of discussion in

philosophy of mathematics. The phenomenon of fluent mathematical reasoning is recognised by many, but it is mostly discussed as a psychological phenomenon rather than as a part of a theory about mathematical knowledge. In order to analyse this phenomenon philosophically, the preliminary concept of intentionality, an important idea in the philosophy of mind, is introduced below.

4.5.1 *On Intentionality*

“... St. Peter could do what he intended not to, without changing his mind, and yet do it intentionally.” (Anscombe, 1957, p94)

Some time ago, a Y9 pupil of mine, Adam, produced the sum of squares formula for his homework on a scrap of paper. There was no explanation or even evidence of experimentation, just the formula. To my question 'why no reasoning?' he replied 'I found it on the bus'. The ambiguity of his reply was amusing (at the time!) and I relate the incident now because it draws attention to the potential intentional disposition Adam had on his bus ride. Was he attracted by a fluttering scrap of paper - and then acted with intentionality in retrieving it? Or was he intending to think up a formula while the crowded bus careered round the Oxford streets, a physical situation in which one would not contemplate writing?

For an individual to find a mathematical formula, which is new to him, requires *intention*, a “mental act” involving a decision and an aim. In his introduction to the philosophy of mind, McGinn (1982) conceptualises intention as one of the antecedents to action, together with desire and belief: “The desire provides the point of the action, the belief specifies the means of arriving at the point, and the intention constitutes the resolve to do what is necessary to get to the point.” (p94). A problem of conceiving an intention as a mental act according to, Rorty, (1980), is that this notion of 'mental act' is nothing more than a piece of technical philosophical jargon. He makes his case by pointing out that both intentions and individualistic pain-like phenomena must be lumped together under the 'mental' umbrella and, really, pains and mental projections to potential action (intentions) have nothing in common. However, in her seminal book on intention, Anscombe, (1957), explains why intention is “something whose existence is purely in the sphere of the mind” (p9): it is, briefly,

because intentions can be modified by the individual in a way that expressions of intention, orders and predictions cannot.

The quote from Anscombe, given at the beginning of this section, indicates that there are levels of intention in human actions: our plans merge with what is inevitable when a 'path' is taken. This observation is pertinent for mathematical action. Almost any example involving deductive reasoning would do, for example, referring to the 'Golden Circle' problem of chapter 3, my plan - intention - to find the required ratio involved inevitable 'mathematical practical reasoning' with its formal deductive consequences. Each step I made in the proof could have some 'intention' attributed to it, yet, the intention 'to reason thus', i.e. use Euclidean geometry reasoning, dominates. (c.f. Anscombe, op. cit. pp46-7) Clearly, mistakes of performance are possible even when the judgment is sound: the intention to use Euclidean geometry reasoning is not undermined by failing to execute this reasoning perfectly, (c. f. p57).

Anscombe was originally a classicist and her concept of 'practical reasoning', important in understanding intention, comes from scholarly analyses of Aristotle. Her view is "the notion of 'practical knowledge' can only be understood if we first understand 'practical reasoning' or 'practical syllogism'... [which] was one of Aristotle's best discoveries". This is a form of reasoning, not just to be applied to ethical intentions and ensuing actions, as Anscombe remarks many commentators have done, but also to be applied to quite physical concerns as taking nutrition, (pp57-62). Aristotle often presents this reasoning in standard syllogistic form but rarely gives a verbal conclusion to the practical syllogism: "The conclusion is an action whose point is shewn by the premises, which are now, so to speak, on active service" (p60).

How this mixture of (a) syllogistic necessity due to form, *which is itself part of mathematics* (b) intention in a 'practical' (i.e. here mathematical) problem and (c) the 'bodily movement' (paraphrasing Davidson) involved in this activity might or does constitute mathematical knowledge is the topic of chapter 7. Furthermore, in that chapter, I shall argue, taking a lead from O'Neill's analysis of Davidson, that intentionality when doing mathematics varies from minimal 'resolve' to significant focus and effort. Further preliminaries for III are the contents of the theses expressed

within chapters 5 and 6, as the notions of warrants and objects are, to a certain extent, amalgamated in the notion of ‘automaticity’ which is the main topic in chapter 7.

4.5.2 *Intention in education*

A person’s potential intentions are a function of both nature and nurture. These factors will determine their scope for ‘resolving’ to act. So as teachers, we try to equip students and pupils with possible intentions, which they can exploit given the opportunity for action. This is particularly evident in areas of the curriculum like ‘personal and social education’, where preparation for safe sex or healthy eating can be understood as increasing the students’ possible intentions, the range of actions possible for them. Teachers cannot, of course, guarantee the appropriate intention is realised in the moment of action, (what were the bananas for?), but I would still say that part of the purpose of education is to widen the range of intentions, to increase students’ possibilities for action. And this includes mathematical action. In chapter 7, I argue for a flexibility of intentionality in mathematical action as intrinsic to mathematical knowledge.

4.6 *Summary*

The ‘philosophical preliminaries’, which are the function of this chapter to provide, are epistemological and ontological. To develop ideas about warrants and belief in mathematics, for I, the idea of ‘truth’ is important. This is because the idea of knowledge is generally underpinned by the concept of truth: ‘I know that 97 is prime’ could only be a justified claim if it was true that 97 was prime. The nature of a warrant for belief (or knowledge) is affected by the conception of truth that is adopted, which is why a discussion of ‘truth’ is preliminary to one on ‘warrant’. The other main preliminary, for II, is ontology: what are the possibilities for describing or delineating what there is which makes up mathematics. Do the propositions of mathematics have solely linguistic referents or are there extra-linguistic aspects of mathematics? Prime numbers, for example, can be thought of as properties of arrays of objects which cannot be placed in non-trivial rectangular form. Alternatively, the concept of a prime numbers can be thought of as a linguistic item within a particular discourse. These questions are from mainstream philosophy of mathematics. The idea of whether fluent

Chapter 4: Philosophical Preliminaries

action can be thought of reasonably as knowledge, relies upon the ideas of warrant and mathematical object developed as well as the notion of intention.

5. Chapter 5: Mathematical Warrants

“On arriving at $(x + 1)^2 = x^2 + 2x + 1$ a pupil said: ‘That’s fantastic: to find something that is true for all the numbers there are’ (Sawyer, 1992)

5.1 Introduction

In this chapter, I pose the question ‘When is a belief mathematical?’. I argue that holding a belief about a mathematical proposition is not sufficient to claim that that belief is a ‘mathematical belief’, for this belief may not be warranted mathematically. This challenges the notion that knowledge is justified true belief: a student may be able to assert justified true belief about a mathematical proposition but not ‘have (mathematical) knowledge’. In this chapter, the first part of the overall thesis is developed:

Learning mathematics involves a change to warranting belief of mathematical propositions by mathematical warrants. These include logico-deductive or quasi-empirical frameworks, rather than empirical or authoritative ones.

This should not seem a radical statement. However, in the context of teaching and learning, I show that there are theoretical issues about the desired type of warranting.

5.1.1 Justification is part of mathematics

Development of beliefs about mathematical propositions, and justification of those beliefs, is at the heart of teaching and learning mathematics. The question of belief is vital for teachers and students. When a student learns a new topic, she does not swallow knowledge as a pill, but, often tentatively, assents to, then later perhaps justifies, propositions about this new topic. Even at the weakest level, giving assent to a proposition, e.g. ‘ $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges’ involves holding a belief. How you justify what you assert or assent to is important in all learning, and in mathematics, justification is also part of the substance of the discipline: a student's proof is both her argument and her result.

5.1.2 Outline of the chapter

Firstly, in 5.2, I look at differences between knowledge and belief and the crucial issue of justification. Then I turn to the epistemological theory, in 5.3, in particular, Alvin Goldman's work. I consider aspects of his theory of 'reliablism' and how it may relate to developing mathematical knowledge. Paul Moser's analysis of 'meta-justifications' is used to assay Goldman's contribution. Goldman's theory is used as he argues for a close connection between epistemology and cognition. Moser's work delves into meta-warrants; this is relevant because the epistemologically privileged status of, for example, 'proof' should be explained. This theme is taken up in the next section where mathematical warrants are further characterised. Then I turn to mathematics specifically and present a selection of notions of justification in mathematics, in 5.4. This is followed by educational evidence, in 5.5, principally from Hanna, and Coe and Ruthven, which deals with student belief causation, and justification. These writers discuss the position of proof in school mathematics particularly. Data from my students are used to illustrate the thesis. The final section, 5.6, relates the philosophical and educational issues further.

5.1.3 Exemplification of thesis in educational contexts

The key point of this chapter is this: there are different rationales one might give for asserting, or assenting to, a belief in a mathematical proposition. While there may be several reasonable justifications for a given belief, there is a distinguished class of 'mathematical warrants'. Only when the belief's justification is given via such a warrant may the belief constitute mathematical knowledge. That is, when a student acquires a new-for-him belief about a proposition of or part of mathematics, the justification that he employs is as much mathematical knowledge as the belief itself. For example, there is a well known algorithm for finding out whether a given multi-digit number is divisible by three. There are various ways that an individual could employ to justify her belief in the efficacy of the algorithm. Indeed, Y9 students (Rodd, 1997) could distinguish between their functional competence ('it works like this:') and their theoretical rationale for why the algorithm works (an algebraic explanation).

On the other hand, a negative example was reported by Lee (1994). She found that the same epistemic subjects assented to the mathematical proposition: “ $0.999...=1$ ” after attending a lecture that included this result, but when interviewed subsequently, while recalling the authoritative answer, indicated their lack of conviction through such phrases as

“ ‘it is infinitely close but not equal to’, ‘there was a page and a half of arguments to show they were exactly equal, but still ..part of me said “no they’re not”’ and ‘[yes]...but, I can’t picture it”’ (Lee 1994, p 131).

In this case her students were not able to justify the proposition in question using a *mathematical* warrant. This reluctant type of assent to the existence of limits will be familiar to any teacher of this topic, and points to the issue of justifications for belief. The ‘choice’ of warrant for the belief that the epistemic subject employs, (implicitly or explicitly), is part of their mathematical knowledge development.

5.2 *Knowledge and belief.*

The ‘person in the street’ would probably say that the job of a mathematics teacher was to teach *knowledge* of mathematics. Why, then, talk about ‘beliefs’ rather than the real aim: knowledge? An answer to this rhetorical question can be given on two levels: Firstly and simply, being able to distinguish in the classroom between a student holding a ‘belief’ and one having ‘knowledge’ is hardly within a teacher’s ken. Secondly, as Plato showed in the Theaetetus, a useful definition of ‘knowledge’ is elusive. Knowledge is more than the sum of its constituents, and every component of any putative definition itself requires definition:

Socrates And it is utterly silly, when we are looking for a definition of knowledge, to say it is right opinion with knowledge, whether of difference or of anything else whatsoever. So neither perception, Theaetetus, nor true opinion, nor reason or explanation combined with true opinion could be knowledge.
(trans. Fowler, 1921, p255)

The last phrase of the quotation above can be paraphrased as: ‘knowledge is justified true belief’, which, although it has some promise as a ‘definition’ of knowledge, under

closer inspection prompts us to ask questions like “what is ‘justification’?” and “how is ‘truth’ secured?” In this way, we can get caught, either in circular definings, (‘truth is that which is justified’), or in an infinite regression (‘a justification is required for the justification of...’).

Nevertheless, this aphorism can serve as a good starting place for investigation, for it draws attention to those essential concepts ‘justification’ and ‘truth’. Not only is justification prerequisite for any knowledge claim, but it is through justification that tentative beliefs, like ‘I think, the limit is zero’ can be strengthened by reasoning, ‘the limit is zero, because J’, where J stands for some warrant. Knowledge, then, at least requires beliefs that are justified. The kind and quality of those *justifications* determine whether mathematical beliefs move towards mathematical knowledge.

However, some philosophers want to distinguish strongly between knowledge and belief. Santas (1995), for example, interprets Hume’s theory of knowledge as making such a distinction: “knowledge, in the strict sense, is a term that for Hume is reserved for those ideas that are demonstrated a priori ... [based] on the logical inseparability of ideas” and Santas develops his theory that education is essentially about warranting belief. I do not want to reserve the word ‘knowledge’ for a sanitised a priori knowing only, for it seems that ‘knowledge’ is set up as unobtainable by the very definitional constraints²⁷. In other words, I do not want to follow Santas’s interpretation that “knowledge and belief are different animals” but place them both within the same species.

In standard Western philosophy texts (e.g. Quine and Ullian, 1970) questions about the nature of knowledge, are made more manageable by expressing what is potentially knowable in a propositional form. Truth values can be associated to these propositions. This is can be seen to be a technically useful device, as such knowledge requires truth and the truth of propositions is simply given by the associated truth value. (Of course, this truth value may be difficult to find out.) For the secondary school mathematics teacher, much of the curriculum can be expressed in this

propositional form: ‘there is a limit to this sequence of numbers’, ‘the sum of the angles of a planar triangle is half a turn’, ‘the 4th decimal place of π is 5’, etc. The issue of the distinction between propositional and non-propositional knowledge is considered in chapter 7, but here the focus is on propositional knowledge.

5.2.1 What is a ‘warrant’?

The word warrant has already been used in context, above, but requires further elucidation. To do this I shall start with an educational context: Quite a common complaint from students around the Y10 (age 14 -15 years) stage is that they can execute the formal motions of elementary symbolic algebra but it holds no meaning for them. For example, consider this exercise:

Take any three consecutive numbers. Square the middle one, and form the product of the other two. what do you notice?

A colleague related an instance of a student who was able to do the algebra, i.e. she was able to form n^2 and $(n-1)(n+1) = (n^2-1)$, respectively but claimed she ‘did not understand’ and was not able to deduce that the numbers differed by 1. Indeed, this sentiment echoes Shazdah, a Y10 student, who said of solving elementary equations “I know what to do but not what it means!”. Going through the motions of a proof does not guarantee the understanding of those ‘motions’ and how they connect premiss with conclusion.

Further insight into the meaning of ‘warrant’, may be found in considering the grammar of the term, i.e., how the word ‘warrant’ is used. A warrant for your arrest necessitates your going to the police station! There is no argument at this stage, the machinery of the warrant takes over. Another ordinary language usage can be found in domestic appliances’ ‘warranties’. These guarantee the appliance’s function, if your washing machine is not functioning qua washing machine, it will be mended, and its

²⁷ Kitcher (1984), whose work is discussed in chapter 6, argues for the impossibility of mathematical a priori knowledge.

existence and function merged once more. There is a suggestion of necessity in both of these which I see as important in the meaning of 'warrant'.

The epistemologist Plantinga opens the second volume of his trilogy on warrant with:

“What is knowledge? More exactly, What is it that distinguishes knowledge from mere true belief? What is this elusive quality or quantity enough of which, together with truth and belief, is sufficient for knowledge? Call that quantity, whatever it is 'warrant'.” (1993, p v)

Goldman, (1986, p2), also refers to “epistemic concepts such as 'knowledge', 'warrant', 'rationality'”. Further on he groups 'justified', 'warranted' and 'rational' together as evaluative terms, but, unlike Plantinga, does not go on to use 'warrant' as a key term or to attempt a definition. Plantinga uses the term 'warrant' in delineating the related concept of 'justification': “justification strictly so-called is no-where nearly *sufficient* for warrant.²⁸ I also argued ... that justification isn't *necessary* for warrant either.” (1993, p vii). Warrants, after all, according to Plantinga, are knowledge-guaranteeing, and it may be possible to 'know' without being able to justify this knowledge. Nevertheless, the focus here is on justifications for mathematical propositions and mathematical warrants.

To an epistemologist who was not interested in focusing on mathematical or genetic issues, mathematics might seem to provide a specific type of warrant in its logio-deductive reasoning. To such an epistemologist, mathematical knowledge is an easy case (perhaps *the* easy case!) of warranted knowledge. For example Goldman asserts “a proof of a mathematical proposition is a necessary component of anyone's being justified in believing it” (1986, p 269). I disagree that it is that simple! What the connections are between justifications, beliefs, warrants and knowledge for the specific domain of learning mathematics, is the subject matter of this chapter.

²⁸ This is exemplified by Gettier style problems, which I relate to mathematics education in 5.3.2.

5.3 *Epistemological Theory*

The way in which we take on beliefs varies. A proposition can start to be meaningful because of a perception, advice from another, or a deduction, for example. Mathematical propositions are no exception to this. What I want to assert is that, however the initial belief about a mathematical proposition is acquired, a mathematical warrant is required before the proposition can be claimed as mathematical knowledge, even though such a belief may be claimed to be some other sort of knowledge. For example, some elementary geometrical propositions are often known empirically before they are known as deductions from axioms. So, propositions of this sort may require a re-justification of the belief, which, in turn, may involve contesting rationales for the belief. The difficulty, for the individual (or a group), to relinquish one form of justification for another is surely considerable. Without explicit advocacy of, or help in, making such a change in the way truths are justified, only exceptional ‘natural mathematicians’ would make such a transition from, in this example, perceptual geometry to deductive geometry. An implication for teaching is that mathematical ideas or propositions introduced or acquired through ‘non-mathematical’ means, may have to be re-justified if they are to count as mathematical knowledge. In practice, people resist their trusted means to understanding being disturbed. This may help explain why learning and teaching abstract mathematics is a difficult task.

There are two aspects of knowledge development which it is desirable to distinguish: the initial belief where the proposition becomes imbued with meaning for the individual and the - for want of a better term - ‘mature’ belief which can be justified by the individual. To take the case of the division by three algorithm: a learner’s initial meaning may have been induced by his dividing several numbers by three, dividing the sum of the digits of those numbers by three, and being asked to remark upon a difference between numbers divisible by three and those not divisible by three. The learner becomes acquainted with this ‘trick which seems to work’²⁹. A mathematical

²⁹ For example, here is a fragment from an interview with one of my Y9 students, Katrin.

justification of the algorithm requires analysis of the decimal system. In this example, the 'distance' between the experimentation and the structural explanation is less than that between empirical geometry and axiomatic geometry. This is because the subject needs to be able to understand place value before dividing is possible for him. But, once he is happy that, e.g., 87 is $8 \times 10 + 7 =$

$8 \times 9 + 8 \times 1 + 7 = 8 \times 9 + 8 + 7 = 8 \times 9 + 15 = 8 \times 3 \times 3 + 5 \times 3$, he has basically got the general structural explanation. In elementary Euclidean geometry, there are deductive proofs of 'perceptually self-evident' propositions like 'the base angles of an isosceles triangle are equal', the formal proof³⁰ of which is much further from understanding what the proposition means than a perceptual appraisal of isosceles triangles.

The standard epistemological theories about how knowledge is acquired categorise routes to knowledge as 'rationalist' or 'empiricist' (see Grayling, 1996, p 39). This categorisation is exemplified by saying that mathematics and logic are paradigmatic reason-based knowledge, acquired by the former route, and that natural science is paradigmatic empirically-based knowledge, acquired by the latter. How useful are such categorisations for mathematics teaching and learning? Clearly, when any sort of cognitive development is to be considered, there not only cannot be a clean divide between the route to knowledge characterised by 'reason' and that characterised by '(sense)-experience'. This is why I have chosen to look at Goldman's work in particular, where cognitive considerations feed epistemological analysis.

The analysis involved in these theories becomes very abstract. However, as I adopt and adapt some of Goldman's and Moser's ideas, I shall try to apply the general

K: When you add the numbers up that they equal a number divisible by 3. I knew that before but I had forgotten it!

M: That's OK, that's fine. What I want to work on now is 'why'. So, I mean do you have any ideas why this should be?

K: No, not really.

³⁰ If $AB = AC$ in $\triangle ABC$, then $\triangle ABC \cong \triangle ACB$ (s.a.s.) $\Rightarrow \angle ABC = \angle ACB$

theories to the specific issue of the move students of mathematics intend to, or pupils³¹ of mathematics are intended to, make towards mathematical justifications.

5.3.1 Goldman

Initial belief development and mature belief justification are highly relevant to mathematics learning. These are notions dealt with in general in the first part of Goldman's 1986 book. My self-imposed task is to ask (a) what do these general ideas look like when applied to mathematics specifically? (b) does Goldman's general theory help support my thesis on mathematics learning?

5.3.1.1 Reliability of belief causes and belief justifications: Goldman's theory

One of the central areas of epistemology is that of the justification, or otherwise, of doxastic attitudes. . A 'doxastic attitude' - which comes from $\delta\omicron\xi\alpha$ meaning 'belief' - is a technical philosophical term that denotes a position on a 'credibility' dimension that includes: being certain, thinking likely, suspending judgement, doubting, denying, etc.

For propositional knowledge, such as in higher school mathematics, a doxastic attitude generally precedes knowledge of the proposition. For example, the proposition 'the number of permutations on n distinct letters is $n!$ ' needs firstly, to be meaningful to the student in terms of vocabulary, notation and purpose, before the student can reasonably be said to 'know' the proposition. I should make it clear, perhaps, that I do consider propositions such as these knowable. My interest is not to probe this assertion on 'knowability', but to get a handle on the transition from initial meaning-making to knowledge. Philosophically, this transition may be considered as a kind of 'epistemic progression'.

This progression includes the stages of developing a *doxastic attitude* towards the proposition, where the meaning of the terms develops. Then this develops into a

³¹ Although I am not quite consistent in use, I understand a distinction between 'student' and 'pupil'. Students choose to study, their intention is to make progress in their chosen field; their initial

positive *belief* in the proposition. And the aim is to form a justification for the belief which, arguably, depending on the justification, constitutes the student's *knowledge* of the proposition's truth.

The sort of justifications that allow this progression to knowledge is part of the subject matter of Goldman's theory. His theory is of the 'psychologistic epistemological' type, which means that contributions from empirical cognitive science are used to develop non-empirical ideas about the theory of knowledge. For this reason, such a theory is of potential interest to exploring the thesis that 'in learning mathematics students change to 'mathematical' justifications'. It concerns an interplay between empirical cognition and non-empirical epistemology. A key assertion of Goldman's theory is that for a belief to count as knowledge it must have been caused by a "basic cognitive process" (Goldman, op. cit. p 93) that is *justified*. And, in this theory, justification itself involves the notion of *reliablism*, which, in turn, depends on a *realist* theory of truth (p 17).

5.3.1.2 *Initial beliefs have causes*

Goldman talks about beliefs being caused by a reliable processes. While this seems reasonable enough, there is clearly work to be done on capturing what 'reliable' might include. He makes distinctions between 'generally' and 'counterfactually' reliable. Not surprisingly: "To qualify as knowledge, a true belief must result from a generally reliable process, not just one that is (counterfactually) reliable for the case in question." (1986, p47). Stepping back, for a moment, Goldman does not explicitly discuss progression on the credence axis, although he recognises that dimension (p14). His emphasis is to characterise positive doxastic attitudes, i.e. beliefs rather than, say, doubts. Goldman's ideas about 'reliablism' are an attempt to explain why a given doxastic attitude is a belief rather than (say) a suspension of judgement. And, crucially, what further is required for that doxastic position to be a knowledgeable one.

relationship is with the subject of study. Pupils are initially in a relationship with a school or a particular teacher; what they learn is a function of that relationship, which may well equip them for further study.

In 5.5.2, I shall describe a typical doxastic situation before starting trigonometry. In that scenario, it is clear that the children will have many cognitive beliefs before they start their new topic, and so much of the teaching work could be conceived to be developing their justifications for their beliefs, i.e. working on getting them knowledgeable.

5.3.1.3 *Beliefs to knowledge: discrimination and reliability*

In Goldman's 1976 paper it "says (roughly) that a true belief fails to be knowledge if there are any *relevant alternative situations* in which the proposition p would be false, but the process used would cause S to believe p anyway. If there are such relevant alternatives, then the utilised process cannot *discriminate* the truth of p from them; so S does not know." (italics in original, quoted from 1986, p46). This is because "[a] sense of 'know' is ... to distinguish one thing from another. ... The conjecture is that the propositional sense of 'know' is related to this underlying meaning, in that knowing that p involves discriminating the truth of p from relevant alternatives" (op. cit. p47) What might such processes look like in mathematics education? The difficulty is that, strictly, mathematical propositions are not contingent, so the sort of counter examples Goldman fabricates (pp 45-6) are not really adaptable to mathematics. Nevertheless, I think that there are situations where p is true, a subject S believes that p , but S cannot discriminate between the truth of p and the falsity of an alternative proposition q , which differs from p slightly. As a teacher, I use something like this criterion for knowledge when, for example, I want to test understanding of functions. I am aware that some students rely heavily, and sometimes uncritically, on the images given by their graphic calculators, so I try to set a question³² which requires this Goldman-type discrimination. Here is an example:

What is the difference, or relationship, between the graphs of these two functions:

① $y = e^{-x}$ and ② $y = e^{-x}(7-x)^2$?

³² Dylan Wiliam has an discrimination example on similar lines.

There are invariably students who say ② is a transformation of ①, for example a stretch parallel to the y axis or some sort of translation, for the image they see on their screen does not indicate the ‘bump’. In this case, proposition p could be ‘for some domain, $[a, b]$, there is a transformation which (nearly) maps ② on to ①’. A corresponding alternative proposition, q , could be ‘for all domains $[a, b]$, there is a transformation which (nearly) maps ② on to ①’.

Another question which requires this sort of discrimination is to ask students to evaluate $\int_{-1}^1 \frac{1}{x^2} dx$. In this case many hand-held calculators return an error message, but the CAS DERIVE returns -2.

These questions can, of course, return a false positive! A student may be sufficiently adept with the technology to make the requisite discrimination but still not have a clue about functions outside what they can get to on the screen!

Goldman does not often consider mathematics specifically. One question he does pose on mathematics is the following: “How should we handle true beliefs in necessities [like mathematical propositions] that do not qualify as knowledge? My suspicion that they are mostly cases in which the belief forming processes, or methods, are not globally, or generally reliable” (op. cit. p 48). The issue of a ‘true belief in a necessity not counting as knowledge’ is obviously central to my theme, here. However, the only ‘unreliable process’ he explicitly mentions is the rhetorically hopeless ‘blind faith’: “Using blind faith to acquire algorithms is notoriously unreliable; an algorithm so acquired cannot transmit knowledge, though the algorithm itself may be perfectly reliable” (op. cit. p 52). He does not work on what are explicitly reliable processes for any field of knowledge. As I have said before, reliable processes for one area of human understanding are not always similarly reliable in their use for understanding mathematics.

5.3.1.4 Beliefs to knowledge: justification

Turning now from Goldman’s views on reliability to his ideas about justification: he chooses to “approach justification in terms of a rule framework” (p 59) because each

justification is not an isolated argument. The rule framework that he proposes is encapsulated in the following principle:

“S’s believing p at t is justified if and only if

(a) S’s believing p at t is permitted by a right system of justificational rules, and

(b) this permission is not undermined by S’s cognitive state at t .” (p 63)

Condition (b) is a disclaimer, but the defining condition (a) involves the loaded terms ‘right’ and ‘justificatory’ which themselves require justification!

In order to see how the application of this theory will go, let us consider two mathematical propositions:

(i) The sum of two odd numbers is even.

(ii) The product of two odd numbers is odd

Justification rules that could ‘permit’ these include rules of infant ‘naïve set theory’, for example, rules for Matching and rules for making Unions. In this case, the subject Simon, having represented a generic example of two odd numbers, matches pairs from each and the odd ones out from both sets form a pair themselves. In the second case, despite a superficial similarity, there is more mathematical complexity, as multiplication is involved. Nevertheless, an analogous infant activity would be to set out a grid of pebbles representing the multiplication, and then to remove pairs until the odd one remains. At the mathematical level of active symbolism they are very similar propositions with similar proofs relying on the symbolic conception of odd as $2N+1$ and even as $2M$:

$$(I) (2n + 1) + (2m + 1) = 2(n + m + 1)$$

$$(II) (2n + 1)(2m + 1) = 2(2nm + n + m) + 1$$

This example points out that there are likely to be more than one “right system[s] of justificational rules” even for elementary mathematical propositions. For, in thinking about what theory is desirable for knowledge in general, the ‘infant naïve set theory

activity' should have a place as "a right system of justificational rules" in a system that acknowledges cognition.

So far, two 'right systems of J-rules' have been suggested that would justify the belief of at least one of these two propositions: (1) 'infant naïve set theory' based on manipulation of material objects and (2) algebraic symbolic manipulation, as exemplified by (I) and (II). Are they both legitimate belief justifiers, i.e. in Goldman's terms are they 'right'? Goldman notes that *both* a criterion of rightness is required *and then* a way of determining whether a particular system of justificational rules (J-rules) satisfies the chosen criterion. So the question just posed has to be postponed until the 'rightness' criterion is established. At this juncture it is also worth posing the falsifying position: what systems of J-rules that might be attempted to be used on these propositions are not 'right'? Although bizarre J-rule systems can be concocted (as "only on Tuesdays" (p 60)), interesting cases are borderline: Specifically, how does the system of J-rules characterised by 'observation' (rather than manipulation) fare? In this case, the observation might be of a data base of pairs of odd numbers together with their sums and products. In terms of Goldman's criterion of justifiedness, given as follows:

"A J-rule system is right if and only if it permits certain (basic) psychological processes, and the instantiation of these processes would result in a truth ratio of beliefs that meets some specified high threshold (greater than .50)" (p 106)

it seems that such an observation system should be acceptable. However, this poses problems for mathematical justifications as the number of observations will necessarily be finite, while the mathematical proposition ranges over an infinite set. To be more blunt:

- ★ Can Goldman's theory exclude, at some level, empirical justifications such as these from mathematical propositions?

Before this question can be answered, some other details in the theory require presentation.

Goldman argues that logic - paradigm of legitimate transition making - cannot itself be all the J-rules, for the reason that a “valid inference pattern...does not require the cognizer to understand why the inference is valid or the connection between the premises and conclusion.” (p 84-5) Analogously, in the odd and even example, the symbolic ‘proof’ requires digestion; gazing at it or writing out the symbols does not make it ‘true for you’. J-rules are to be thought of as cognitive processes - within the domain of psychology rather than logic. “What makes for rightness is not just a matter for logic; it is also a matter of psychology. After all, *modus ponens* is no more *valid* than other valid argument forms. It is just psychologically simpler than others.” (p 89). This explicit demand for psychology within these J-rules is balanced with an equally explicit demand for its absence in the ‘rightness’ criterion: The criterion of rightness should only include non-epistemic terms (p 63). In other words, to avoid self-referentiality, this crucial criterion of what are appropriate-ways-to-transform-belief-to-knowledge (i.e. J-rules) must not employ notions that it sets out to explain; it is free from ‘belief’ and ‘knowledge’. In short, he proposes a non-epistemic criterion for a ‘right’ system of justifying, but the particular rules of which - i.e. the means by which cognitive transitions are made - are psychological.

On this criterion of rightness, Goldman’s position is one of “non-relativistic pluralism” (p 70) - there are more than one system of J-rules that can be considered right. The ‘rightness criteria’ that he considers, but rejects, include ‘logical’, ‘social’, ‘coherent’, and ‘evidential’. What he comes up with is that “a case can be made for *mixed* conceptions of justifiedness” (p 73). In other words this non-epistemic aspect is subject to a choice factor. This seems to push the problem to another level, for should we not then require a ‘right’ way to decide which ‘right justificatory rules’ were to be used to justify a belief?

However, Goldman’s theory cannot distinguish ‘mathematical’ warrants from other classes. Indeed, *learning* mathematics, as opposed, perhaps, to being a fully fledged mathematician, certainly involves different systems of J-rules: object manipulation, observation and symbol manipulation, as well as others, including social, justificatory frameworks, all have their role in establishing beliefs about mathematical propositions, as any teacher knows. It does not seem that Goldman’s theory can make the sort of distinction indicated by ★. This is because, on this level, it is not clear how

to decide which of the plurality of conceptions of justifiedness is applicable. What Goldman does offer is a theory that distinguishes between epistemic status of the rules we use and the rationale for using those rules. But in order to find a way to make distinctions about the relative merit of certain J-rule systems as they pertain to mathematics learning, I shall turn to Paul Moser, 5.3.3, whose 1993 book 'Philosophy After Objectivity: making sense in perspective' includes further analysis of warrants. In particular he investigates what warrants the warrants: he develops a theory of meta-warrant.

Goldman makes theoretical distinctions within the notions of reliability and justifiedness which have some applicability in explaining and supporting my thesis here: there are initial doxastic attitudes which may have been caused by reliable methods or otherwise; conceptualising justifications as being underpinned by criteria of rightness, lends weight to the idea that learners of mathematics need to be discriminating in their use of justifications. But, exactly how these 'reliable processes' are recognised, exploited and communicated is not within his thesis; the specifics for mathematics learning are to be located in the domain of mathematics education.

5.3.2 *Gettier problems.*

Related to the ideas of Goldman is the issue of 'Gettier-problems', (Gettier, 1963), for these concern the applicability of a justification. A Gettier-type problem is of the following type: a true belief has a justifiable warrant, but the warrant is misapplied. Examples of Gettier problems in the literature, for some reason, often involve automobile ownership! For example: Fred believes that Jane has a Morris car. This belief is true, Jane does have a Morris car. Fred's belief that Jane has a Morris is justified by several sightings of Jane driving a certain Morris. However, that Morris belongs to Jane's mother. So Fred's belief is true and justified, but the justification is not justification for the actual original statement's truth: 'Jane has a Morris car'. The point is that most people would be uncomfortable asserting that Fred had knowledge of this item, even though he had justified true belief.

Analogously, within mathematical reasoning, there is ample opportunity for Gettier-style attribution. For example, Andy believes that the limit of the sequence ' $1/n$ ' is

zero. This belief is true, the limit of ' $1/n$ ' is zero. This belief is justified, he thinks, by 'several sightings' of many terms of the sequence. The justification, like the observations above, is not foolproof. Just as the empirical observation of seeing someone drive around in a certain car does not imply ownership, so seeing a sequence get closer and closer to zero does not mean that it does have a limit and that that limit is zero.

So there are two sorts of issue concerned with beliefs and limits of sequences. Firstly, the student's easily taken-on belief that the limit is the number that (approximately) pops out after a large number of terms of the sequence have been calculated, but they still do not believe that the limit actually exists; it is never reached. Secondly, the Gettier type problem: the student might attribute knowledge to his belief that the limit of ' $1/n$ ' is zero, because of his warrant: 'I have worked it out to thousands of places several times'. This is a case where we might be uncomfortable to attribute knowledge to the student, as his reasonable warrant for empirical propositions is not the best one with which to *justify* a mathematical proposition.

5.3.3 Moser's theory

To deal with the meta-epistemological issues of the adequacy of justificatory standards, I shall refer, briefly, to Moser, 1993, (esp. pp 60 -105). He expands a thesis that is neither 'god's-eye objectivist' nor relativist and I am interested in pursuing arguments that develop a middle way. He points out "Notions of justification ... admit of evaluation, at least relative to...certain conceptual purposes" (Moser, 1993, p 13); the purpose in this case is deciding which set of (in Goldman's terms) J-rules are justified for justifying mathematical propositions especially for those involved in developing mathematical beliefs (either their own, as learners, or those of others, as teachers).

I felt Moser's book to be technical to the point of obscurity. His key points seem to be that neither realism nor idealism (for which I have used the term 'anti-realist' above) can escape their own 'question-begging'. For example, to assert that "conceiving-independent identities" exist because I can see them (p5) begs the question against realists, (an analogous statement, like 'conceiving-independent entities do not exist

because seeing, touching, etc. does not ensure their existence' begs the question against anti-realists). His main thesis is to assert "agnosticism is our best option from an evidential point of view" (p 6). If this was the only result in his book, reading it would not be worth the struggle! For only the extraordinarily dogmatic would not allow a tinge of agnosticism into their ontological world view! The problem which I hoped that Moser's work would help to elucidate is expressed in the question 'how to justify which justification justifies?'

Moser addresses this question of justificatory standards through the device of 'semantic foundationalism'. This should, he claims, avoid the naïveté or circularity to which, he explains (pp63-75), the question of standards can often reduce. A key notion he introduces is that of 'operative constitutive standards'; i.e. that justificatory standards are 'purpose-relative'. Let us consider the idea of 'purpose relative' justification in the domain of mathematical propositions, in particular, consider the odd and even scenario sketched in 5.5.1.3, does this analysis of Moser's privilege the manipulation over the symbolism or vice versa? There are two levels here - I think that these are what Moser calls 'epistemic' and 'non-epistemic'. The first is *finding out that* an odd plus an odd is an even, the second is *showing that* an odd plus an odd is an even. The first is non-epistemic, a conjecture from perception, the second is epistemic, it involves a justification³³. Each has a fitness for purpose. Is there anything from Moser's theory which will motivate a transition from the non-epistemic to the epistemic? Not that I have yet been able to see.

Despite the promise which a theory of meta justification seemed to hold, I do not find Moser's work sensitive to the developmental transitions which are so much part of educational concerns. He does clarify some issues in epistemology (like the "semantic...explanatory [and] evaluative project[s]" (p 60-1)). However, in terms of the application which I am seeking - what justifies a re-justification? - the notion of 'purpose-relative' itself begs the question! For why should I prove what I already feel

³³ Moser's use of 'epistemic' and 'non-epistemic' refer to whether or not *knowledge* is involved, with its attendant requirements of truth and justification. This is different from the distinction I made in chapter 4, between 'epistemic' and 'psychological', because the latter is a scientific, rather than philosophical, term. Moser's 'non-epistemic' is related to the formation of a doxastic attitude.

I know (e.g. two odd numbers added are even)? The answer that the proof is suitable for mathematics has a strange 'should' tone to it (c.f. Moser and 'Jones' p 99). On the other hand, if a symbolic, general, proof *is* the only way to know this as mathematics, then we still have the epistemological gap between science (properties of objects of our perception) and mathematics (properties of generalities or relations).

So this foray into abstract epistemological analysis does not really clarify the notion of mathematical warrant. Nor has it explained, in a non-self-referential way, what characterises truly mathematical warrants for knowledge, rather than justifications for mathematical propositions. It has helped to clarify distinctions between knowledge (by virtue of a warrant) and various forms of belief.

There are alternative tacks I could have taken: for example, I have not reported on, described or analysed a priori approaches to mathematical knowledge. This is because of my interest in how mathematical knowledge develops. As Kitcher (1984) remarks about a priori 'privilege'

"Frequently an apsychohistoric epistemology is developed by attributing to some propositions a special status..[which] have the property of counting automatically as items of knowledge if [the subject believes them]. Once apsychohistoric epistemology has granted to some axioms of mathematics this privileged epistemic role, the question of how we know these axioms disappears and one who raises that question can be accused of dabbling in those psychological mysteries from which twentieth century epistemology has liberated itself. " (p 14).

And, of course, like Kitcher, I do not want liberation from thinking about the psychological mysteries! What I shall turn to now is the particulars of mathematical warrants, applicable for novices, rather than follow Kitcher's more general treatment, in trying to present the case for distinctive mathematical warrants and their necessity in the acquisition of mathematical knowledge.

5.4 *What are mathematical warrants?*

To say that mathematical knowledge requires mathematical warrants is clearly circular unless some other way to distinguish mathematical warrants can be given. A substantial answer to the question of the nature of a mathematical warrant requires replies from various perspectives: For example, study of mathematical foundations could give insight into the nature of mathematical deduction, study of the sociology of mathematicians could suggest why certain people's arguments were respected over others' and study of mathematical history might show how standards of proof have changed over time. A full response to this question is, therefore, beyond the scope of this present work. What I shall do is restrict myself to the educational context of higher school mathematics and present some ways of justifying belief in mathematical propositions which I shall argue can be knowledge conferring mathematical warrants (for an appropriate epistemic subject).

Proof is the paradigm mathematical justification 'as we all know', but in what sense proof warrants is less clear. This is the subject matter of 5.4.2. Other examples of justification that I claim may be a mathematical warrant are quasi-empiricism (Lakatos) and visualisation (Giaquinto). Furthermore, the relationship of warrant to action could be said to be a 'procedural' mathematical warrant is discussed in chapter 7, but I shall make reference to it in this chapter on occasion. I also observe that justifications that are sometimes used in the classroom, like empirically testing cases or accepting an authority are not mathematical warrants, although they may serve sometimes as pedagogical devices to help the pupil develop some initial belief.

The pedagogical point is that non-mathematical warranting of a belief in an ostensibly mathematical proposition does not result in a student's having mathematical knowledge; the warrant must be mathematical in the case of a knowledge claim. Yes, this does mean that for knowledge claims we must be able to override our initial, say perceptual, warrants for a belief by a mathematical one, which is why I now discuss what a mathematical warrant might be. This shows why it is useful to distinguish between the psychological and the philosophical. For, once I have a belief in stable memory, it does not make sense to re-cause that belief, but it can make sense to re-justify or warrant it.

An example of warrant and non-warrant

To put some flesh on the bones of these abstractions, I shall exemplify a situation of a non-warranting justification and a corresponding mathematical warrant.

There are several ways of thinking about this³⁴ number pattern:

1.3.6.10.15. ...

There are different names associated with some of these different ways of thinking. For example, triangular numbers, sum of the first n integers, an arithmetic progression, steps numbers, a quadratic formula. There are also many ways of justifying belief in various propositions to do with this sequence. As I related that Moser suggests, there is a sense that warrants are relative to the proposition concerned. And it is for such a reason that student progress to the use of mathematical warrants may be psychologically difficult, as the student may be familiar with the mathematical object (1,3,6,10,...), have justified belief about propositions concerning it, but not see why s/he has to believe another proposition about the sequence in a different manner. I have seen this problem in practice in the classroom in the following case: a student has a concrete or diagrammatic representation of the sequence, s/he observes that the next term in the number sequence is obtained by adding on one more than was added on before. This generalisation is surely justified procedurally by the action of adding on one. If the student needs to know whether 153 or 15,083,778 are in the sequence, my claim is that the procedural-justification is no warrant. Even if the membership, or otherwise, of 153 could be checked procedurally, the likelihood of error in the 15 million-plus case would militate against this procedure being a warrant for the belief in the proposition '15,083,778 is in the sequence'. (The question of running a program to check this with a minimum of error requires an epistemic relationship with the aid being used, which is a digression here.) In the case of checking whether a given number, T , is in the sequence, one could set up the equation

³⁴Of course there are an infinite number of sequences that start thus

$$\frac{n}{2}(n+1) = T$$

and solve the quadratic. Clearly, this requires knowledge of the formula for the sum of the first n integers as well as a functional competence with solving quadratics.

What is a *mathematical* warrant for the belief that the sum of the first n integers is half the product of n with $n+1$?

I was a sixth-former (16 or 17 years old) the first time I ‘proved’ $\sum_{k=1}^n k = \frac{n}{2}(n+1)$. It was a proof-by-induction, easily executed, poorly understood. The purpose of being asked to do the proof was not to do with belief in the proposition but with practice in executing the process. So in this case, a proof was not a warrant for belief in the proposition, even though, such a proof could be correctly considered a mathematical warrant in general. To answer the question ‘Why is the formula for the n^{th} term thus?’ a warrant must include the generality of the result with an epistemic connection to the result. The following ‘informal’ proof is more likely to establish this connection: a visual image of two copies of the steps representation of this number sequence of indeterminate sizes are ‘jig-sawed’ together to make a rectangle of width n and height $n+1$. Hence the formula for one of these blocks of steps must be half the area of the rectangle, i.e. half of the product of n with $n+1$. Giaquinto (1989, 1992) argues that such ‘visualisation’ can be epistemic in the sense that it is knowledge producing (see section 5.4.5 below).

Here are three candidates (there are doubtless more) for a method to show that the sum of the first n integers is half the product of n with $n+1$: (a) counting-on procedure; (b) induction-proof; and (c) visualisation. Which, if any, is knowledge producing? To recap my observations: (a) the count-on does not illuminate the structure - it will not produce knowledge although it may strengthen the subject's belief; (b) the induction is so smooth the content of the proposition may slip away - it depends on the subject whether this will justify as well as warrant; (c) the visualisation is particular, structure illuminating - but whether this image can be reinterpreted in terms of the numbers from whence it came should be a test of whether it constitutes a warrant; perhaps it will just be a static image.

Warrant may be seen as a successful justification. Justifications are, of course, commonly conceived as proofs. Clarifying the relationship of warrant to proof in mathematics education is part of this thesis. However, there is a substantial body of literature on proof and mathematics education and it is not my concern either to review this literature or to add to it, although I do use the work of some authors in this chapter. While I shall not explicitly review the mathematics education literature on proof, it would be as well to mark its domain: Since ancient times proof has, of course, been part of mathematics, but the discipline of mathematics education is a more recent phenomenon. Within this discipline, discussions of proof have often been related to the cognitive aspects in learners' proofs and the affective issues of proof in the curriculum.

A fairly early mathematics education study was that of Bell (1976). This reported on a survey of students ideas about proof. Bell also conceptualised: justification (that), illumination (why) and systematisation (how) aspects of proof. While I would expect Bell to concur with my point about the 'Teflon' nature of proof structures like that of induction', (part of) his thesis rests with this notion that it is not proof which secures knowledge. This is in distinction to mine, for I want to assert that mathematical warrants ('proof', whatever that means, being paradigmatic) are the only way to have strictly mathematical knowledge. A particular consequence for a teacher, is not to pretend that students' mathematical conviction can be achieved by employing his or her charisma.

As I write, there has been a resurgence of interest in mathematical proof at school level, at least in Britain. Recent editions of the professional magazines have given over entire editions to this issue (Mathematics in Schools, June 1996 no. 155, Mathematics Teaching Nov. 1994, Vol. 23 no.5). This follows an edition of Educational Studies in Mathematics focused on 'Aspects of Proof' in 1993, (Vol. 24, no. 4). A brief explanation of this current flurry is that the 1960s' overly formal curriculum (see 5.4.2), at least in North America and Britain, was rejected. And proof, construed as formal proof, was also excised from the school mathematics curriculum. There is now a lot of activity to try to reclaim the baby (proof) so hastily thrown out with the bath water of formalism! This quest is not confined to practising teachers (the target audience for the professional magazines) but extends to mathematics education

researchers. The last ICME conference produced a hefty document, with many international contributions, on the topic of mathematical proof in infant to undergraduate curriculums (de Villiers, ed., 1996). Many of the contributions in this collection were reports of actual student behaviour or attitudes to mathematical proof. My thesis here should contribute to the understanding of why it is that proof is so important in mathematics learning: it is because without mathematical warrants, the students' knowledge of facts about mathematics and procedures with mathematical concepts does not amount to mathematical knowledge.

5.4.1.1 A task with which to explore warrants:

Take a mathematical proposition you know to be true. Find three or more justifications of this proposition (analogous to the sum of the first n integers case). Which of your justifications warrants?

5.4.2 Hanna

Gila Hanna's work, over many years, has dealt with the notion of proof in mathematics education. She has criticised excessive formal proving in mathematics in education as well as advocating that students learn about mathematical proof. I shall use work of hers from the extremes of this period to investigate the relationship between proof and warrant in mathematics education. In particular, I use Hanna's 1983 argument against formal proofs and her 1995 argument for proofs of various kinds, to support my thesis that, for mathematical knowledge, belief is not necessarily warranted by a 'rigorous' proof, but, (as expected!), a less restrictive notion of proof is a fundamental mathematical warrant.

5.4.2.1 Rigorous proof, per se, is no warrant.

When Goldman asserts that "a proof of a mathematical proposition is a necessary component of anyone's being justified in believing it" (Goldman, 1986, p 269), as quoted above, what does he mean by 'proof'? If Goldman means a formal proof, then Hanna's thesis will refute his assertion. For Hanna claims that mathematical meaning - which is necessary for any doxastic attitude - does not arise from a formal proof per se. Hanna argues that rigorous, or formal, proof, (where the axiom system is specified

and the steps of the proof are explicit, (Hanna, 1983, p 3)) is far from what mathematicians use in their practice to justify a belief in mathematical propositions³⁵. Hanna distinguishes between formal, or rigorous, proofs and informal ones (ibid. p 67). She defines informal proofs as the complement of rigorous proofs in the class of proofs and exemplifies the sort of explicit proof that students of 'new math' in 1960's North American high schools would have been expected to learn (e.g. p 22). And so, if, on the other hand, Goldman intends 'proof' to mean an informal proof, then the onus is on him to explain this notion.

For a belief to be warranted, (as I have said before), there must be an epistemic connection between that warrant and the epistemic subject. This is an obvious observation: babies don't know theorems! The difficulty arises in assessing whether the proof-warrant is part of a subject's epistemic tool-kit; in other words: when could a given proof be the warrant for a belief (for a particular person). In section 5.5.2 (in this chapter), the practicalities of this problem are discussed with reference to some 16-18 year old students 'proofs'. At this juncture the issue is treated more theoretically. Hanna (1983) gives an historical overview, as well as detail from individual mathematicians, to show, not only that there is no absolute standard of proof, but also, "a mathematician is still much more interested in the message embodied in the proof than its formal codification and syntax" (Hanna, op. cit., p 72). Proofs, even of the purportedly formal variety, are culturally relative and not (necessarily) meaning-making. The mathematician, Bill Thurston, (Thurston 1994) concurs with Hanna's thesis that it is rarely a proof per se that convinces, for new results, it is the "community standard of proof" that is paramount.

The relationship between mathematics-in-education and mathematics-in-mathematical research is a theme that runs through Hanna's writing. The notion of proof, which is the formal evidence of a researched mathematical result, must have its seed in the school curriculum. The attempt to place the theoretical (and not, in practice achieved)

³⁵ If mathematicians needed proof *before* justified belief, how could they motivate themselves to work for many years on developing a proof? Andrew Wiles reported (BBC Horizon, 1995) that he worked alone for seven years on Fermat's last theorem.

standards of rigor of researchers onto school children led to an unpalatable formal curriculum, as her 1983 study shows.

5.4.3 Simple logical inference as a basic mathematical warrant

Now I want to be explicit about a mathematical warrant; it is all very well noting **how** or that justifications fail to warrant but it is important to specify what **does** warrant! What does warrant is 'perspicuous logical deduction', in particular **modus ponens**. The argument that I shall put forward for this could be interpreted as asserting a **form** of logicism. Be that as it may³⁶, I hope to show that this sort of logicism has an experiential foundation.

If school mathematics and research mathematics are, as I shall assume, in some sense root-to-branch connected, epistemic connections could be given by this thesis overall: connections are warrants, objects and actions. In this discussion of warrant, I make a case for the place of elementary logic, particularly **modus ponens** reasoning ($p \Rightarrow q$ and p then q) to be the seed that connects pre-formal with formal reasoning. Only by establishing a connection between these two, can we make sense of the psychological progression of the individual to the point where proof *does* become a warrant. I do want to make a case for elementary argument to be a warrant.

This claim does not negate Thurston's (ibid.) reports of his own experience and Hanna's (1983) reports of others, that, at the research level, ideas and community support are at least as important as formal proofs for belief formation. An empirical hypothesis is that these mathematicians, who rely on informal communication of ideas and community support, will all have **modus ponens** reasoning available as a warrant for belief.

Rudimentary logic seems to be an innate human capacity related to that of language, and it is this capacity - together with other illusive factors like language and imagination - that support the warrant of proof of this paradigm type. Contemporary

³⁶ Shapiro, 1992, discussing 'Proof and Knowledge in mathematics' opens with "Logicism, as every body knows, is half right"

mathematical research allows, for example, the technically unsurveyable computer assisted proof of the four-color theorem to count as a proof. But, at the initial stage of learning about mathematics and proof, Wang's aphorism that a proof "should be perspicuous, surveyable, or capable of being taken in" (Wang 1962/86 p135) is a pragmatic requirement. At the level of learning, Wang's perspicuous requirement is essential. As a student develops, parts of proofs and other results are remembered, a shared set of meanings and notations are employed with other mathematicians, and the results from mathematicians with good reputations are considered favourably (Thurston, *ibid.*, Hanna, 1983, p 70).

Modus ponens is the mediæval logicians' name for the ancients' formulation of a basic law of syllogistic reasoning, (Glymour, 1992, p 56). It is a formulation of the "container metaphor" (Johnson, 1987, p 23). Johnson uses the term 'metaphor' itself in a metaphorical sense to indicate that language, thought and action are connected in understanding. Modus ponens is an instantiation of the 'container' metaphor because we (sentient beings of human type) understand through our bodily relation to the experienced world that, for example, if the room is in the house and I am in the room then I am in the house. It is for this reason that elementary logic is both a lived experience and a reasoning tool. Of course, it is the authoritative teacher who names it 'logic' and offers symbolic referents, but the child herself needs no name to experience the 'containment'.

I shall now turn to Pólya's discussion of an elementary proof to illustrate

(a) how modus ponens reasoning is a basic proof-warrant of a potentially formalisable type;

(b) that proof warrants of this type, despite their bodily genesis, still require teaching

Consider the proposition that the angle sum of a plane triangle is half a turn. Pólya (1945/90 p 216-7) indicates a proof of the proposition by drawing a diagram of a triangle ABC, constructing a line parallel to the base BC through the vertex A and concluding the result by implicit reference to the alternate angle property of parallel lines. The implicit detail of the argument relies on these modus ponens inferences: (i) if alternate angles are equal, ($p \Rightarrow q$), and each base angle of the triangle is alternate

with an angle at the vertex (p) then these angle are equal (q); (ii) if angles along a line add to half a turn, $(p \Rightarrow q)$, and these angles are along a line, through A, (p) then these angles add to half a turn (q).

Pólya goes on to say:

"If a student has gone through his mathematics classes without having really understood a few proofs like the foregoing one, he is entitled to address a scorching reproach to his school and to his teachers. ... If the student failed to get acquainted with this or that geometric fact, he did not miss so much; he may have little use for such facts in later life. But if he failed to get acquainted with geometric proofs, he missed the best and simplest examples of true evidence and he missed the best opportunity to acquire the idea of strict reasoning. Without this idea, he lacks a true standard with which to compare alleged evidence of all sorts aimed at him in modern life." (Pólya, 1945/90 p217)

Pólya is saying that it is through examples of thinking like this, that our rudimentary modus ponens capacity is set to work on mathematics. Mathematics is an accommodating domain for this sort of reasoning in a way that domains where the container metaphor is not as consistently applicable are not. For example, art appreciation or emotional counselling do not lend themselves in the same clear cut way to this 'if-thenism'. Elementary axiomatic Euclidean geometry is a particularly useful sub-domain as it is possible to have quite short, easily surveyable, proofs that have a persuasive perceptual prop in terms of our (local) Euclidean experience of space and its representation as the proof's diagram. An implication from the quotation from Pólya is that Euclidean geometry was never on the curriculum for students to know about geometric facts themselves, but for students to know about logical reasoning, which, given plausible axioms, can prove the existence of geometric facts.

Hanna's 1983 thesis is directed at refuting the assertions that rigorous proof is part of the (day to day) practice of mathematicians and that there is a common standard of what constitutes a proof. If her thesis is accepted, the consequence that high school students (assuming they should emulate mathematicians), should have this 'rigorous proof' as part of their education, is also refuted. Hanna does not investigate, here,

what positive meaning-making, activities related to 'proof' are, nor the educational consequences of removing 'proof' from the high school curriculum. In her 1995 article, after the removal of proof from the North American, as well as the British school curricula during the interim period, Hanna turns to what proof can offer mathematics students. In this context she widens the notion of proof to include informal -as well as formal - proofs. She notes that proof needs teaching; it is not just the naïve logic from 'containment' (in Johnson's sense). And that methods of teaching formal proof will often include informal proof processes:

"In exploring new ways to teach proof, [experimental] studies have shown the value of such approaches as debating, restructuring, and pre-formal presentation, all of which posit a crucial role for the teacher in helping students to identify the structure of a proof, to present arguments, and to distinguish between correct and incorrect arguments" (1995, p 44)

I have no argument with this outline, but I do want to modify her following assertion which is that the point of a classroom proof is principally to explain:

"While in mathematical practice the main function of proof is justification and verification, its main function in mathematics education is surely that of explanation.

To say that a proof should be explanatory is not to say that it cannot take different forms. It might be a calculation, a visual demonstration, a guided discussion observing proper rules of argumentation, a pre-formal proof, an informal proof, or even a proof that conforms to strict norms of rigour." (1995, p 47)

The case that I have been making here is that a purpose of 'formal-type' proof in the classroom is that this is a reliable way of coming to conclusions. Not just to explain why but also to explain what explanation is, as Pólya said. While modus ponens is at the root of our mathematical explanations it is also a structure within our bodily experience. This is why this sort of elementary logic is both 'concrete' and formal. And permits a transition from the pre-formal to the formal.

To sum up this section: Formal proofs do not of themselves constitute a warrant for belief in mathematical propositions. However, formal proofs are a theoretical standard of the mathematical community. Some formal proofs (at least in a weak sense of 'formal') are perspicuous. The modus ponens argument form is a structure that covers many of these. Modus ponens is an embodied feature of life experience. This means the essential structure of that experience is there for a teacher to employ on mathematical objects, thus establishing a mathematical argument for a novice. This is an explanation for how formal-type proof can begin to become a warrant for belief.

5.4.4 *Quasi-empiricism*

Another possible warrant for mathematical belief is that of *quasi-empiricism* applied to mathematical processes and concepts. Although apparently originally coined by Russell³⁷, quasi-empiricism was championed by Lakatos (1976, 1978) in his extension of Popper's 'Conjectures and Refutations' work to mathematics. Lakatos's motivation in developing his theory of the 'logic of mathematical discovery' was to direct discussion in the philosophy of mathematics from formalists' foundationalism to the issue of how (epistemologically, rather than linguistically or socially) there becomes new mathematical knowledge. As he says in the beginning of 'Proofs and Refutations'

"the purpose of these essays is to approach some problems of the *methodology of mathematics*. I use the word 'methodology' in a sense akin to Polya's and Bernay's 'heuristic' and Popper's 'logic of discovery'...The recent expropriation of the term 'methodology of mathematics' to serve as a synonym for 'metamathematics' has undoubtedly a formalist touch. It indicates that in formalist philosophy of mathematics there is no proper place for methodology qua logic of discovery."

His desire to break new ground in the discussion of the nature of mathematical knowledge between the 'super tight' formalists and the unproductive (in terms of specifying what knowledge might be) skeptics is indicated in his subsequent remarks:

³⁷See Corfield 1997 footnote 33.

“Now this bleak alternative between the rationalism of a machine and the irrationalism of blind guessing does not hold for live mathematics: an investigation of *informal* mathematics will yield a rich situational logic for working mathematicians...which cannot be recognised, still less simulated, by the formalist philosophy..” (Lakatos, 1976, pp3-4).

There are several ‘buzz words’ in mathematics education like ‘fallible’, ‘conjecture’ and ‘quasi-empiricism’, which, while not originally Lakatos’s, were given publicity and credibility by his treatment. ‘Fallible’ (originally from C. S. Pierce, quoted in Putnam, 1990, p7) refers to the nature of mathematics; ‘conjecture’ (borrowed from Popper’s usage) connotes potential fallibilism within premises of mathematical arguments; ‘quasi-empiricism’ suggests the means by which knowledge develops uses empirical, sensory, experience and also employs deductive inferences within this knowledge generation.

Lakatos’s exposition in ‘Proofs and Refutations’ was a ‘rational reconstruction’. Despite the ‘classroom’ scenario, the aim was to describe, by developing an example, the underlying structure of mathematical knowledge increase: these reasoning processes produce knowledge in this way. Thus mathematical ‘knowledge’ is subject to a certain contingency as a result of this method.

Can the theory that Lakatos developed to explain the growth of mathematical knowledge in research-mathematics also be used to justify learners’ new-for-them mathematical knowledge? In other words:

1. Can learners warrant their mathematical beliefs ‘quasi-empirically’?

If this is possible then from a teacher’s point of view it is important to ask:

2. How do learners of mathematics take on quasi-empiricism as part of their ‘epistemic tool-kit’?

After analysing some of Lakatos’s claims, I shall return to these questions in 5.4.4.5.

5.4.4.1 What is quasi-empiricism?

Lakatos was a philosopher of mathematics interested in the practice and detail of mathematical progress. He wanted to introduce the essential informal element of mathematics as it is practiced into a philosophical (a meta-level) explanation of how mathematical knowledge progresses. He had the overarching desire to present an alternative to ‘formalism’³⁸ yet retain mathematics’ deductive character. The product of this ambition is quasi-empiricism:

“Classical epistemology has for two thousand years modelled its ideal of a theory, whether scientific or mathematical, on its conception of Euclidean geometry. The ideal system is a deductive system with an indubitable truth-injection at the top (a finite conjunction of axioms) - so that truth, flowing down from the top through the safe truth-preserving channels of valid inferences, inundates the whole system.

It was a major shock for over-optimistic rationalism that science - in spite of immense efforts - could not be organised in such Euclidean theories. Scientific theories turned out to be organised in deductive systems where the *crucial* truth value injection was *at the bottom* - at a special set of theorems. But *truth* does not flow upwards. The important logical flow in such *quasi-empirical theories* is not the transmission of truth but rather the retransmission of *falsity* - from special theorems at the bottom (‘basic statements’) up towards the set of axioms.”

(Lakatos, 1978, p28, italics in original)

This excerpt explains the structure of quasi-empirical reasoning. It is a kind of deductive inference where, for example, modus tollens, ($p \Rightarrow q$ and $\neg q$ then $\neg p$), rather than modus ponens, is the logical lever and is described as ‘bottom up’ reasoning. Lakatos exemplifies this kind of reasoning throughout ‘Proofs and Refutations’ (Lakatos, 1976) where he illustrates how both

³⁸ Lakatos’s notion of formalism is older and more metaphorical than Hilbert’s Formalism, (see Larvor, 1995, p 88).

global and local counter examples ($\neg qs$) can refute a given hypothesis (i.e., imply a $\neg p$).

While Lakatos' own writings publicised and argued for this radical 'bottom up' structure of the development of mathematical knowledge, he did recognise the Euclidean vein in mathematical practice. This 'top down' reasoning is essentially conservative ("puritanical, antispeculative" 1978, p 29) and so is not the source of anything new, but it has its place:

"We saw that mathematical proofs are essentially of three different types: pre-formal; formal; post-formal. Roughly the first and third prove something about that sometimes clear and empirical, sometimes vague and 'quasi-empirical' stuff, which is the real though rather evasive subject of mathematics. This sort of proof is always liable to some uncertainty on account of hitherto unthought-of possibilities. The second sort of mathematical proof is absolutely reliable; it is a pity that it is not quite certain - although it is approximately certain - what it is reliable about." (1978, p 69)

So Lakatos accepted that mathematical reasoning had a deduction from premisses component, but he relegated it to uninspiring consequentialism and denied that it had a creative role in the development of that which was new³⁹. Speculation and subsequent criticism provides potentially new knowledge. The reasoning that can 'refute' is, nevertheless, still Aristotelian in its traditional logic. What is more Lakatos makes great pains to distinguish the character of mathematics from that of science:

"a theory which is quasi-empirical in my sense may be either empirical or non-empirical in the usual sense: it is empirical only if its basic theorems are spatio-temporally singular basic statements [of science]." (1978, p29, italics in original)

Quasi-empiricism is not induction. Mathematical theorems do not arise from generalisations of observations per se. Nevertheless, observations from mathematical

³⁹ this conclusion is disputed by Corfield, 1997, who claims, on the contrary, that "the appropriate use of rigorous definition has not acted as a hobble on the creativity of mathematicians, but rather an invaluable tool in the forging of new mathematical theories and extension of old ones" p 100

‘play’ can stimulate the sort of conjecture that may become new knowledge. In terms of Lakatos’s theory, does quasi-empirical reasoning constitute a mathematical warrant? Logically, because the reasoning is deductive, the answer cannot be but affirmative. But this reduction to logic is to miss the substance of what Lakatos’s theory has to offer; the point is that it is the “basic theorems” that, in Lakatos’s metaphor, “inject truth” and that they do this essentially by a refutation.

5.4.4.2 Is mathematics quasi-empirical?

In order to apply Lakatos’s theory to learning mathematics, some adaptations need to be made: Lakatos was concerned with theories of mathematics, with the large scale vision adopted by the foundationalists (who failed to justify mathematics in the generality they aimed for). Whether holist or serialist is learning style, novices in mathematics proceed in their knowledge acquisition in relatively small steps. Does (or can) quasi-empiricism warrant their progression?

Perhaps the first point to consider is still at the philosophical, rather than educational level, for do we not have to be convinced first that mathematics is indeed quasi-empirical? Zheng (1990) argues that mathematics is not, in Lakatos’s sense, quasi-empirical for the two reasons (i) that there is not the claimed dichotomy between ‘Euclidean’ and quasi-empirical systems: they are both deductive (as indeed Lakatos said); (ii) informal theories (or the results of informal theories) are to be the potential ‘heuristic falsifiers’ for the formal mathematics, (as Lakatos explains in his 1978 collection, p 36), but “the truth value of statements cannot be injected into the formal theory directly from the corresponding informal theory” (Zheng, p390).

Zheng is putting forward the idea that once a formal theory has been established it is impenetrable. Lakatos tries to counter this criticism in advance by offering a hypothetical example where an informal theorem falsifies a formal one, (1978, p 37). Because the human-powered proof contradicts the result from the formal theory, the sense is that the formal theory hypothesised is not actually about ‘arithmetic’. Hence the formal theory is falsified as applying to this domain. Clearly, there are delicate semantic issues here: when is a formal theory ‘arithmetic’? Nevertheless, a point to note is that the domain of Lakatos’s discourse is on a very high powered mathematical

level. It is crucial to Lakatos's argument that Zheng's criticism does not hold, for it is this informal 'injection' of truth-value from informal theories to refute formal ones that makes mathematics meaningful and deductive.

Whether Zheng's criticism can be made to hold or not depends on the definition of 'formal system'. I do not believe that Lakatos required every 'formal theory' to be in set theoretic form, but, at least such a theory would be axiomatized. Euclidean and non-Euclidean geometries should, therefore be candidates for such a formal system. I want to use geometry to present an example that refutes Zheng's denial of mathematics being quasi-empirical with mathematics of the level of higher school mathematics.

To set up this example, let us consider again the proposition discussed by Pólya: The angle sum of a triangle is half a turn. In section 3.2 the reason for considering this proposition was so that the student might learn, by exemplification, what a basic deductive proof in mathematics is like. Here, I want to exemplify a 'falsifier'. There is such a rich history to the development of geometry that I shall not even start to give it a 'rational reconstruction' historically, (see Gray, 1987), but to attempt, very briefly, a 'rational reconstruction' cognitively:

An heuristic falsifier to the proposition is that a triangle made on the surface of the globe (vertices at Bandung (B), Libreville (L) and the North pole (N), say), has angle sum of about π of a turn. An initial classroom experience for the student is to measure the angles formed by such a triangle on a physical globe. The perception of a triangle with three right angles sets up a cognitive conflict. The standard response, from my teaching experience, is for the students to deny that BLN (with BL, LN and BN drawn on the globe's surface!) is a triangle. However, when they fail to answer questions like: 'what is it then?', and 'how big does a triangle have to be so that it is not a triangle any more?', they (usually) accept the terminology and are in a position to try to explain the result. By explaining the result in terms of the curvature of the sides of the triangle, the students have falsified the original proposition, 'The angle sum of a triangle is half a turn', but have located the 'guilty-lemma': there was an assumption that the triangle was planar (or more strictly that it was a Euclidean triangle).

This example indicates how mathematics, even at the higher school mathematics level - rather than the research mathematician's level - can be quasi-empirical. As Gray's (ibid.) exposition illustrates, counter-examples to existing theory, like this one, in accordance with Lakatos's theory, prompted new research directions in mathematics.

5.4.4.3 *'Naïve abstractionism' and quasi-empiricism*

Is quasi-empiricism 'naïve abstractionism'? The term 'naïve abstractionism' is taken from Gray (1992, and see chapter 6) who uses it to describe the "ontological modes" of mathematicians at the turn of the nineteenth century:

"We must first establish what the prevailing ontological modes were at the start of the century. The most usual was what may be termed 'naïve abstractionism': the idea that mathematics deals with idealisations of familiar objects." (Gray, 1992, p 228)

This is essentially what Zheng offers as a reconciliation in order to retain the label 'quasi-empirical' for, at least, some aspects of mathematics: "what 'quasi-empirical' means here is the indirect relationship between theories and empirical activities rather than the ways of transmission of truth value" (Zheng, op. cit. p391).

My reading of Lakatos is that he has taken pains not go down the 'naïve abstractionist' route. However, it may be the case that 'quasi-empiricism' - so tightly defined by Lakatos - has acquired, in language use, a different connotation which is, essentially, Gray's 'naïve abstraction'. Now, Gray argues that 19th century mathematicians had to undergo a 'revolution' in their thinking and abandon the naïve abstraction that had hitherto stood them in good stead. Results like that of infinite series of continuous functions not converging to a continuous function, were outside the empirical interpretation that Zheng suggests gives justification (in his interpretation of quasi-empiricism). These results required a formalization both to make sense of and to justify. Modern mathematics, then, has required an understanding of 'formality' (used loosely), to make meaning - this is the essence of Corfield's 1997 paper.

One of the difficulties here is in the use of the word 'formal'. On one hand, Lakatos uses the term loosely and disparagingly. On the other hand, Lakatos demands that a

formal theory should be thoroughly set theoretic. His example, cited above from his 1978 collection, page 37, is likely to be ‘formal’ in the loose sense, although - for his argument - it must be ‘informal’ to be able to falsify the ‘computer-generated’ style of formal proof. I believe that Lakatos is open to the criticism that he has allowed a continuum of ‘formality’. The cumulation is the never-to-be-achieved full formal proof, but where is the cut off point that distinguishes informal from formal?

The practical answer to this question is that it depends on the situation. If quasi-empiricism, in Lakatos’s sense, can be a warrant for mathematical belief, then the level of formalization against which (relatively) informal theories can be tested has to be brought down to a more elementary level. As the non-Euclidean geometry example illustrated, the curved-sided triangle falsifies the initial proposition, and stimulates better precision in formalizing/axiomatizing the planar geometry from which the proposition is deducible.

5.4.4.4 *Quasi-empiricism as dialectical*

Larvor’s 1995 thesis explores the dialectical dimension of Lakatos’s work in depth: he quotes Lakatos as having the ambition to “become the founder of a ‘dialectical’ philosophy of mathematics” (Lakatos archives, folder 12.1, quoted in Larvor 1995, p 2). Larvor makes a convincing case that the theory of dialectics - rather than ‘mere’ fallibilism - is central to Lakatos’s understanding of the nature of mathematical progress⁴⁰. Through the dialectical structure, formal structures are both further shaped by criticism and the ‘off-cuts’ from shaping provide stimulus for new questions and directions.

Classroom mathematics can also be ‘dialectical’. The key requirement is that “a ‘proof’ or falsified conjecture can show where to go next” (Larvor, p 92). This dialectically based theory - that in mathematical work, there is an “intrinsic unity between the logic of discovery and the logic of justification” (Lakatos, 1976, p37) - underpinned the ‘investigations’ curriculum in British schools in the 1980’s. In

⁴⁰ the editors of ‘Proofs and Refutations’ suggest (Lakatos 1976, p146, footnote ^{2*}) that this “Heglian background grew weaker as his work progressed.”

practice, ordinary children did not behave like Lakatos's 'pupils' and the freedom to 'investigate' did not, generally, stimulate a generation of children to discover mathematics through their own research. Dialectics, can still be a basis for devising a mathematics curriculum, but, nowadays, there is no longer the expectation that ordinary pupils will spontaneously engage in mathematical research 'at their level'. The teacher can introduce a dialectical progression through the classroom materials she offers, but, of course, this does not imply that any given pupil will mirror this progression in his thinking.

Lakatos's 'Proofs and refutations' classroom dialogue is hypothetical. It is designed to illustrate the dialectical theory he was propounding by exemplifying, for example, uses of counter example to refute or hone mathematical propositions. Larvor notes that we can read 'Proofs and refutations' on three levels:

1. The level of classroom interaction. The 'pupils' conduct their own research. This allows the reader to follow the logic of a mathematical investigation;
2. The level of a seminar in Pólya-style heuristics. Within the 'pupils' dialogue, specific advances are made through heuristical reasoning (analogy, specialization) and so these forms of reasoning are communicated in situ.;
3. The level of a philosophical investigation. What is the nature of a sound proof? How are conjectures improved? The need to answer these questions comes from actually doing mathematics. (see Larvor, op. cit. p 86)

The process of criticising the 'pupils' proof of the Euler formula at various stages does give insight into the nature of what it is to mathematically convince. This idea of 'conviction' is intimately connected with warrant. It is to the warrants of learners of mathematics that I now turn.

5.4.4.5 Quasi-empiricism and the classroom

Can quasi-empiricism be a mathematical warrant for actual students? In 5.4.4.2, I gave an example, at the higher school mathematics level, that was intended to show that mathematics could be quasi-empirical. Now, to return to the questions posed at the beginning of 5.4.4:

1. Can learners warrant their mathematical beliefs ‘quasi-empirically’?

If this is possible then from a teacher’s point of view it is important to ask:

2. How do learners of mathematics take on quasi-empiricism as part of their ‘epistemic tool-kit’?

I shall take the second question first. The basic logical structure of quasi-empiricism is that of modus tollens. And the most basic instantiation of modus tollens is a counter-example to a conjecture: Let p be the conjecture and q a deductive consequence; if $\neg q$ is discovered, then, by modus tollens, $\neg p$.

To grasp the logic of quasi-empiricism, a pupil has to be really convinced that one counter-example can ruin a whole theory! ‘Ruining’ a theory that has been obtained by the pupil has greater psychological effect than refuting some theory into which little psychological investment has been placed. Many mathematical problems English and Welsh children do for the Ma1 requirement of their National Curriculum include devising a formula for a number pattern they have been unearthing from a geometric or numeric context. The following problem, when interpreted by a pupil as an exploratory task, may help them appreciate the finality of a counterexample.

Take a whole number, square it, add the number and add 41. Is your answer a prime number? Example: take the whole number 3: $3 \times 3 + 3 + 41 = 53$, a prime!

A pupil with a facility in algebra would not need to explicitly test various numbers to deny the question. But for those without this skill, repeated application of this procedure, may have conjectured that it does return a prime. In such a case, observing that 41×43 is the result of applying the procedure to 41, should effectively squash this hypothesis. But more than that, it should help to reinforce that one counterexample can ‘ruin a whole theory’.

To turn to the first of the questions posed: Can learners warrant their mathematical beliefs ‘quasi-empirically’?

Once the structural reasoning of modus tollens is in place, this logic can be used to test procedures as well as - in the case of counter-examples - propositions. Let us consider the following activity I have used in several classrooms both Y8 or Y9 and Y12:

Can you solve this equation: $2^x = 100$? What about this one: $x^x = 1000$?

For the younger students, the purpose of setting this exercise might be to practice decimal place-value; to introduce them to features of a calculator, for example, its 'power button' or its graphing facility; to experience the process of 'trial and improvement'. For the older students the activity is an introduction to logarithms.

The younger students will not be equipped with a mathematical warrant for their belief in their numerical result (that, in the first case $x \approx 6.6439$, in the second $x \approx 4.55556$). Their result is but an empirical one that relies heavily on the functioning of their calculator. Their work might result in a 'primitive conjecture', but they need to ask themselves 'How do we know we have the solution?' and 'Why is it that number?'; there are no reasons behind the answer other than 'I checked it and it was close'. With the theory of functional inverses in general and the relation between logarithms and exponentials in particular, the Y12 student is able to solve the first of these equations directly (with tables or a calculator). Whatever the precision of the result required, it will just depend on the calculation $\frac{2}{\log_{10} 2}$. The theory of logarithms

serves as the 'formal theory' that has been 'axiomatized' after the initial empirical 'play' in Y8 or Y9. Now the student tries to apply this formal theory to the next question: $x^x = 1000$. The result of 'proceeding as before' is to find that $x \log_{10} x = 3$, but x cannot be extracted from this equation so easily. This 'refutation' of the 'taking logs' procedure, both confirms the 'formal theory' and starts to delineate the boundaries of that theory's applicability, ('these are the sort of equations this procedure works for'). I suggest that this is an example of Lakatosian quasi-empiricism at the higher school mathematics level. In this way, a student's belief in $\frac{2}{\log_{10} 2}$ being the solution to the first question is warranted by the combination of the formal theory of logarithms together with a notion of where this is not applicable.

I conclude that Lakatos's theory of quasi-empiricism can constitute a mathematical warrant for those learning mathematics.

5.4.5 Can visualisation be a mathematical warrant?

This section draws on the work of Giaquinto (1989, 1992) who argues that visualisation can be construed as epistemic: it can be knowledge producing. His meaning of 'visualisation' means more than sight-perception, it also connotes this warrant for certain mathematical beliefs. Giaquinto argues for 'visualization' being a source of mathematical discovery. An implication of accepting his proposal is that 'visualizing' can be a mathematical warrant. Giaquinto emphasises that visualizing and proving are distinct. It is because of this difference, that I want to explore the claim that visualizing can constitute a mathematical warrant, itself and not just as a psychological motivator for mathematicians.

5.4.5.1 What is Giaquinto's visualizing?

Giaquinto develops both his description of visualizing and his argument for visualizing by considering a simple geometric configuration and asking a question about a property. The configuration in question may be constructed as follows: construct a square and the midpoints of its sides. Join the midpoints to form another 'middle' square. The property is that there are now two squares, one bigger than the other. The question he asks is 'How much bigger is the original square to the middle square?'

Giaquinto's emphasis in this paper is 'How can the answer 'twice' be justified in an "epistemically acceptable way"?' His answer constitutes his claim "that visualizing can be a means of discovering a geometrical truth" (p 384). In other words, his 'visualizing' can constitute a warrant for mathematical belief. Giaquinto makes his case with reference to the example. Specifically, he claims that the belief is justified through visualization if the following hold:

- "(a) one feels that a future counterexample is not even an epistemic possibility;
- (b) the putative evidence of sense experience is meagre at best;

(c) one believes that the putative evidence is of a kind which could not warrant [such a] belief "(p 388)...

"(d) the phenomenology of looking and noticing is absent;

(e) one has a feeling of certainty in [the proposition] which is not undermined by recognising the fallibility of inner observation" (p 391)

This is what Giaquinto-visualizing is. Whether or not it is 'visualization' in the ordinary language sense is not important.

What is important is whether there is, in Tahta's terms a "geometric imperative" (1989) that can be used to "discover a truth by coming to believe it in an epistemically acceptable way" (Giaquinto, p382). In Tahta's article, the notion of 'geometric imperative' could be summed up by his phrase: "One cannot not do geometry" (Tahta, op. cit. p 27). If Tahta's term is used in a sense considerably narrower than in his 1989 article, then I think that what Giaquinto is trying to get to is a conceptualization of a 'geometric imperative' which is a-linguistic. Tahta's article gives many examples of a-linguistic geometric experience; it does not concern itself with the epistemic issues which are the focus here. A rephrasing of Giaquinto's enterprise, then, is 'assessing the epistemic acceptability of the geometric imperative'. A feature that Giaquinto points to is the intentionality of visualizing which is not present in seeing and which 'internalises' the geometric imperative.

The visual aspect of mathematics is most important and has received attention from mathematicians, as well as educationalists (like Tahta, among many) and philosophers. A mathematician that has written reflectively on this visual aspect is Philip J. Davis (Davis, 1993). Davis uses the term 'visual theorem' to connote a wide range of mathematical propositions which can be believed visually. His emphasis is more on the mathematical proposition than the process of visualization; he does not investigate the nature of visualization, but finds examples of mathematical propositions the truth-value of which can be (presumably, in his experience) apprehended through (what he considers) visualization. His point is that some geometric truths are (naturally, in the sense of instinctively) comprehended visually.

He asks provocatively: "do you think that vision can be totally algorithmized, co-opted by [symbolic] mathematics?"

Davis offers several examples of 'visual theorems', one of which is the following:

In a circle C of radius 1, draw a smaller circle C_1 contained in C . Draw a circle C_2 located in C and not overlapping C_1 . Draw C_3 in C overlapping neither C_1 nor C_2 .

Can you keep drawing circles 'forever'?

For Davis, it "should be visually obvious that the infinite series $\sum r_n^2$ is convergent" (where r_n is the radius of circle C_n). He then says that it is "by no means obvious" that the series $\sum r_n$ also converges, but encourages the reader to "determine the answer and a way of looking at things so that it becomes obvious". In other words, use 'visualization as a means to discover a geometric truth'.⁴¹

5.4.5.2 Anecdotal evidence of some of these requirements

Wanting to have a sense of a fresh view on the familiar configuration of Giaquinto's geometric example, I decided to get my young son's response. Because I wanted to communicate with him, I used a cut out square with the middle square marked; he could not have processed a verbal description. I also realised that the language of proportion, "How much bigger?", was inappropriate, so I asked him which was bigger 'all the outside triangles together or the square in the middle, or are they the same?'. His direct response was 'the square's bigger'. I then asked him to fold each of the triangles over their line (that was an edge of the middle square). He tried to get his fold along the line, but not being very dextrous did not manage a very neat job. Nevertheless, he then looked up and said 'they're the same'.

⁴¹ I 'see' $\sum r_n$ represented by a spiral - which could be topologically dense in C - from the centre of C never quite getting to the boundary. So the length of this line is potentially infinite.

What is the relationship between this young child's physical manipulation, and the mental-manipulation that Giaquinto calls visualizing? Clearly, they are not the same. The mental case includes calling up the possibilities-for-action in a way that the actual manipulation of material cannot. It is difficult to assess whether the child did recognise that it did not matter that the folding over was not exact, or whether he could not distinguish between exact or non exact folding (i.e., between approximate and precise equality). It is the case that he interpreted his action in such a way as to transform his hypothesis, (à la Popperian falsification, perhaps).

My other child was also subjected to 'the example' (this time, in Giaquinto's language). She is a competent Y10 person, but did not get the answer straight away. She needed to draw a diagram. On drawing her diagram, she saw (visualized) the middle square as the 'square on the hypotenuse' of a corner triangle. By specialising the length of the original square to 10 units, the area of the middle square was calculated using Pythagoras's theorem as 50 sq. units, clearly half of the original square. As soon as she had done this she said 'what a prat I've been!' and indicated her belief in the certainty of the proposition by, what was essentially, the Giaquinto visualization.

The responses of both these novices can help understand the meaning of some of the requirements for visualization given above in 5.4.5.1. In particular, the meaning of (d), which initially can seem paradoxical, is explained: The younger child physically manipulated paper to stimulate this visualization, the elder one drew a diagram, but in both cases there was a catastrophic 'falling into place' of the spatial concepts and their consequences. It is also interesting to note that for the older child, the initial visualization (seeing with intention) was the Pythagorean icon. This is a legitimate visualization from the diagram, but it was not until she knew the answer, that the big square was twice the middle square, did she re-vizualise the situation. The clarification of intention resulted in the more direct visualization.

5.4.5.3 Visualization includes mental-manipulation

The process of Giaquinto-visualizing is not just (mentally) seeing, but, crucially, mentally manipulating. Giaquinto certainly emphasises manipulation, but seems to conflate seeing and doing in the 'visualization' package. The example he focuses on

does not seem to require a sighted epistemic subject, a square can be felt and so can the folds and their (near) meeting. Action in our imagination on 'ideal objects' 'abstracted' from actual sensory experience is what he seems to be describing. However, I think he would eschew this description because the 'scare quoted' terms are contentious.

Giaquinto tries to get round this problem of incorporating action, by introducing a 'concept group of transformations'; the transformations idealise actions, like folding, and are inextricably linked with the concept itself (squares and diagonals, for example). He justifies incorporating these ideas into his argument by giving them the status of data - they come from "empirical research in mathematics education" (p 395). However, what he uses as evidence, is not so much empirical as itself theoretical: a network of constructs is developed to help those interested in learning mathematics conceptualise the visualizing process. My point is that Vinner and Hershkowitz have come up with ideas which may shed light on the same data, or phenomena, but their theory does not properly confirm the philosophical theory. Nevertheless, the data on which they both rest, can refute either. As an aside, it is curious that Giaquinto conceptualises the 'folding' as a rotation, (this is clearly incorrect - visualise a non-isosceles triangle!). Why should reflections not also be in the 'concept group'? I think that the introduction of such constructs, which rely on a mathematical-metaphorical element, (like 'group') deflects from the issue of explaining the basic phenomenon of 'visualization as a means of geometrical discovery'.

5.4.5.4 Visualizing and time

One of the features that Giaquinto stresses for visualizing is its immediacy, (p 394). Is this tautological or consequential? Neither! My view is that to associate any temporality with visualization is incorrect. I believe that this view can be justified by reference to right and left brain functions. Just because, in the case of his example, he immediately visualizes the folding corner triangles covering the middle square exactly, it does not mean that a geometric imperative is instantaneous. Time is actually irrelevant in visualization; for together with logic and language, time is a left brain concept. Any attempt to visualize and be aware of time will, in practice, resort to

a linguistic mode of thinking. Betty Edwards book 'Drawing on the Right Side of the Brain' (Edwards, 1979) sets up experiences for understanding that statement which do not really have a verbal explanation. In other words, visualizing is a-temporal rather than immediate.⁴²

I want to consider an example that (i) indicates the possible a-temporality of visualizing, and that (ii) convinces in the case of a geometric proposition. The example is a play on the one offered by Giaquinto:

"How much bigger" is the area of a regular pentagram than the area of its central pentagon? Twice as big? Less than twice as big? More than twice as big?

In this case, to divert a digression into trigonometry, a choice of possibilities is offered. The result was not immediate to me (as Giaquinto's example was, which is why I needed some input from novices in that case), but, by using 'the same' folding-visualization the result becomes certain⁴³. Using Giaquinto's criteria for visualisation, the question of the comparison of the size of the whole pentagram with its central pentagon, is answered as follows:

Compare these isosceles triangles on a common base: one triangle is one of the five pentagram's star-petals and the other triangle is a fifth of the central pentagon (this triangle's vertex is the centre of the figure), respectively. When a pair of these triangles is folded along their common base

- (a) it is impossible to imagine them as other than unequal - so a counterexample is not possible;
- (b) it does not matter how dreadful a diagram I draw or how rough a model I make, the triangle from the pointed part of the pentagram is longer and therefore (because of

⁴² While a critic would claim that I am doing the very thing that I admonished Giaquinto for doing, viz. relying on other theories, I would like to counter that I am using other data. In particular, I use personal data obtained from working at the exercises in Edwards's book.

⁴³ Some teachers would call this a 'geometric proof' and Lakatos would have classed it as an 'informal proof'.

their common base) larger than the triangle from the central pentagon part - so sense experience *does not* augment the evidence;

(c) no measurement would be able to influence this comparison - so sense-experience sort of evidence *would not* make any difference to the knowledge-status of the proposition;

(d) the action of folding or visualizing folding is all that is needed to know they are unequal. No further scrutiny is required. - So no further visual inspection will yield more relevant information;

(e) the certainty I feel is to do with the necessary unequalness of the triangles - so this is not undermined by my recognition that my inner observations are fallible.

I understood and contemplated the pentagram question well before I experienced the visualization and yet, to satisfy (i), once the visualization was made, the truth of the proposition was not a worry. How are visualizations 'thrust upon' one? Certainly, I needed to have the concept of this geometric shape familiar, including the crucial property that the angle at the vertex of the point's triangle was less sharp than, (actually half of), the corresponding angle of the other triangle, before this geometric fact could be visualized.

5.4.5.5 *'Inner experiments' are not visualizations*

The distinction between 'inner experiments' and visualisations is a difficult one to make. I think it is right to try to make it for the following reasons (i) the 'eureka!' quality of a correct visualization is a psychological phenomenon (see Hadamard (1945) - not that all 'eurekas' are visual) and, as Giaquinto remarks, this phenomenon should be incorporated into an account of mathematical discovery (ii) visualization allows an ability to idealise, that the meaning of 'experiment' does not normally entail. The problem that I have is that inner experimenting frequently precedes the illumination of visualization, (sometimes by years - as I have said, time is an irrelevance once the problem is part of one's subconscious). Indeed it is difficult to anticipate how visualizing can take place without some sort of inner experiment - for surely intentionality implies an element of enquiry which is, essentially, experimental?

Experimental results can be in error, and Giaquinto does not claim either that visualizing is exempt. He classifies four types of visualizing error (p 398-9):

1. failure to visualize what one intends to visualize
2. acquiring a false belief about what has been visualized
3. believing a general truth from an insufficiently general visualization
4. "vagueness and inconstancy of the images" (p 399)

Of these, 4 is a disclaimer and not developed; (I am not convinced that it is distinct from 1). The description of errors 1 - 3 sheds further light on what it is that visualization should be, or rather, indicates what a slippery concept it is.

I do not believe that there is a robust distinction between inner experiment and visualization. Experiments incorporate the feature of scrutiny and subsequent noticing of something; visualizations are intention-directed geometric presentations. For this reason I tentatively accept them as knowledge - not 'just' belief - producing.

5.4.6 A visualisation (for the experience and the post-experience reflection)

As a post-script to the discussion on whether visualisation can warrant I offer this visualisation. The task is (a) to try to visualise the situation presented below in words; then (b) to assay whether the visualisation has potential for producing knowledge.

DESGARGUE'S THEOREM: Take three distinct circles. For each of the three pairs of circles draw the pairs of common tangents; let q , r , and s be the points where the respective pairs of tangents meet. Then, q , r , and s are co-linear.

Visualisation⁴⁴: Take three (generally unequal) spheres, Q , R , and S , and place them on a plane. Take another plane and rest it on 'top' of the three spheres. This plane (miraculously) cuts through the other plane; it cuts through in a straight line. The sphere R touches each plane once in r_t and r_b , say, and the lines $r_t s_t$ and $r_b s_b$ meet at q

⁴⁴ This was told to me by a Cherwell School student, David Norland.

(and correspondingly for Q and S). The points q , r , and s are collinear because they all lie on the line of intersection of the planes.

Although this visualisation seems three dimensional and the original problem was two dimensional, just project the three dimensional version into the bottom plane. For generic points of projection the two dimensional result holds.

5.5 Belief and its genesis in learning mathematics

Belief precedes knowledge. And a belief requires a warrant to be classed as knowledge. This is the epistemological conceptualisation that I have been developing in this chapter's previous sections. I have, in particular, been arguing that, for mathematical knowledge, 'justified true belief' in a mathematical proposition is not enough. For the belief in a mathematical proposition to be considered as mathematical knowledge that belief must be justified by a mathematical warrant. While there is an obvious circularity about explaining 'mathematical warrant' in terms of mathematical practice, the plain state of affairs is that we are trapped within our own discourse. There is no truly 'meta-warrant' (the foundationalists' failure to 'globally' formalise is testimony to that). I have presented various means of justifying which may serve as mathematical warrants - basic logic, quasi-empiricism and visualisation - and assessed them in this regard. The distinction between the notion of a 'cause of a belief' and that of a 'justification for a belief' has also been noted. In line with Goldman's ideas about the intimacy of cognition and epistemology, these notions are not neatly separable. As a trivial example, I might justify a belief by some reference to an authority whence the belief originally came. Nevertheless, I'd say that *belief causes* are principally cognitive (within the domain of psychology) and *belief justifications* are principally conceptual (within the domain of philosophy).

5.5.1 Towards analysing student belief

I now want to turn to classroom specifics. According to the conceptualisation of mathematical belief and knowledge which I have been developing, mathematics teaching includes educating students

(a) to recognise a mathematical justification - for which they need to be introduced to specifically mathematical warrants

(b) to be aware of how (i.e., by which warrant) their belief in a given proposition (or in the efficacy of a given procedure) is growing

If these two requirements were realised, a student should be able to move relatively easily from his/her initial warrant in a belief, to a mathematical warrant. I have observed, over many years work with students of all ages, that their connection with their mathematical beliefs is often subject to the 'mother duck' syndrome: the first 'sighting' of some sort of justification is the one that remains with them.

5.5.2 Some types of student belief

Beginning a new topic in mathematics involves forming new beliefs. Progress towards knowledge is made when those beliefs are justified. They become knowledge when those beliefs are warranted. To flesh out the meaning of this statement, I want to consider the topic of trigonometry - which is new for secondary school pupils - and suggest some specific beliefs learning this topic involves. The typical learner, in this situation is between the ages of 12 and 14 years (in the UK anyway). These children will already have beliefs formed about triangles and angles and other concepts that are involved in trigonometry. I shall not attempt to delve further back in their cognitive history, but to start, as a teacher has to, by making some initial assumptions about 'what they should know'.

A list of beliefs that are pertinent to early trigonometry is given below. (No claim is being made for this list's uniqueness or completion). All the statements of belief concern right angled triangles:

1 - the side opposite the right angle is called the hypotenuse

2 - the hypotenuse is the longest side

3 - if you mark one of the smaller angles, call it α , then α is formed by the hypotenuse and another edge of the triangle

Chapter 5: Mathematical warrants

4 - this other edge is called the adjacent side to α

5 - the side of the triangle that does not help form α is 'opposite' this angle

6 - if the angles of the triangle are fixed, then the ratios of pairs of sides of the triangle are the same no matter what the size of the triangle

7 - these ratios have special names, for example, ' $\tan \alpha$ ' is the name for 'opposite/adjacent'

8 - $\sin \alpha$ and $\cos \alpha$ are always less than 1 ($0 < \alpha < \text{a quarter turn}$)

9 - if α and the hypotenuse are given as actual numbers, you can work out (inter alia) the opposite side's length using the formula: $\text{opposite} = \text{hypotenuse} \times \sin \alpha$

10- $\tan \alpha = \sin \alpha / \cos \alpha$

A classification of these propositions can be made according to how each proposition might be justified by a pupil:

(a) information (1,4,5,7): the belief could come from an authority, in which case the justification is authoritative;

(b) consequence (2,3,6,8,10): the belief could come from a deduction, in which case the justification is logical;

(c) perceived (2,6,8): the belief could come from sense-data, in which case the justification is empirical

(d) operational (3,9): the belief could be related to action, in which case the justification is procedural.

Clearly, for some students, 2, 6 and 8 can be 'consequential', I have placed them in the 'perceptual' category too as this could be the belief forming mechanism.

This classification draws attention to different sorts of justification and requires consideration of which of these constitute mathematical warrants. In 5.4.4.2, I discussed the mathematical character of deduction. I shall develop the notion of

procedural warrants further in Chapter 7, but this 'operational' category could be akin to Vergnaud's 'théorème-en-acte' (Vergnaud, 1981) or a sort of procedural warrant the like of which I discuss in the later chapter. By marking these categories explicitly, the limited function of empirical and authoritative justification is apparent.

Nevertheless, the way a given proposition is justified by a given student is, of course, personal. In the example above, a student may justify 2 by noting that the hypotenuse is opposite the largest angle, a deductive consequence, or by measuring many hypotenuses and observing the data (easily done with Cabri or similar dynamic geometry package), an empirical consequence.

Proposition 6 in the list can be used to emphasise the distinction between the *cause* of the belief and the *justification* of the belief. The initial belief may have been stimulated by an empirical investigation given several similar right triangles (as the activity in the text book SMP 11-16, Y2 p 22). The 'justification' of the belief does not *have* to be given by a corresponding perceptual-empirical reason - 'I measured them and this is what I got' - but could be justified, for example, 'procedurally' by subsequent enlargements or deductively, by recourse to similarity.

5.5.2.1 *Belief development*

Mathematics has both 'process' and 'content' components. The relationship between these aspects of mathematics has been the focus of several people's research in recent years. Sfard (1991), for example, considers these components 'dual' and uses "different sides of the same coin" to indicate their mutual dependence. These ideas are developed in Sfard's subsequent work, e.g., with Linchevski (Sfard and Linchevski, 1994). In another school of mathematics educational thought, Tall and colleagues have used the all encompassing term 'procept' to denote this 'duality': "a procept [is] a process which is symbolised by the same name as the product" (Tall, 1991, p 254). In terms of tracing the development of belief, the point is that a conceptual justification at one level can serve as a cognitive cause of belief at another.

To exemplify this, consider the mathematical proposition " $1/7$, expressed as a decimal, repeats." Suppose that Brian believes this proposition because his calculator

displays 0.142857142 in response to the key sequence 7 x^{-1} enter; the cause of the belief is his trust in the functioning of the machine (an authoritative warrant) together with what he reads on its display (a perceptual-empirical warrant).

$$\begin{array}{r} 0.14285714 \\ 7 \overline{)1.00000000} \end{array}$$

If Brian were able to do the division $7 \overline{)1.00000000}$, and recognise that the sequence of remainders from the divisions repeated, and would repeat indefinitely because of the very process of executing the division, then I would say that this would be a justification of the belief (using a procedural warrant) - and a mathematical one at that! Indeed, once such a mathematical warrant has been used as justification, this enactive competency makes it hard to appreciate the tentative nature of the previously used belief warrants. Furthermore, to elucidate my point above, the explicit division justification feeds (cognitive) causes for beliefs at the level of ‘if I work out $1/n$ as a decimal by dividing I can only get up to $n-1$ remainders before it starts repeating’ etc. So the belief formation process is entwined with the process of justifying beliefs, and hence the justificatory warrants⁴⁵.

5.5.3 *Data from students' work: some student justifications of the angle in a semi-circle property*

As a basic aim of a mathematics teacher is to help her students gain mathematical knowledge, and if it is accepted that at least part of mathematical knowledge consists of mathematically warranted true belief of mathematical propositions, then it behoves a mathematics teacher to seek to detect the use of such warrants. In the following discussion, in which I analyse students' work, I recognise that all I am able to do is interpret the data rather than offer an argument for a proposition. My interpretation of

⁴⁵ This is Lewy's point he makes in Wittgenstein's lecture: "Suppose we take:

$$1:7=0.142857142...$$

have I shown that these [emphasised] figures *must* come here when I've done the division simply? Or have I shown that they must come there only when I've *proved* the recurrence?

... *Lewy*: By dividing, you've shown that those figures must come there." (Wittgenstein, 1976, p120-1)

the data suggests that there are robust differences between students' epistemic functioning characterised by whether or not they employ mathematical warrants in their public reasoning.

A class of first year prospective primary school mathematics specialists was given the following question as part of their first assignment during their first half term at HE college:

Prove that the angle in a semi-circle is a right angle

(i) using vectors (ii) without using vectors.

Devise another way of explaining that the angle in a semi-circle is a right angle.

This question was designed to elicit their distinguishing between a mathematical fact and a mathematical argument. The photocopied work of two out of the 36 students is in Appendix 9.1; these two were chosen to exhibit representative responses from the class. I want to discuss the answers given to part (i) and (ii) by students Emma and Joanne, because their solutions indicate different ideas about what constitutes a mathematical explanation. (To what extent their 'public' presentation of proof reflects their 'internal' epistemic connection to the proposition they were asserting, can, of course, only be surmised.) The reason why a three part question was set, was to give a multi-faced opportunity for the students to establish their belief of the proposition. In other words, opportunity for different justifications were encouraged, including, at this stage, empirical ones. I was quite receptive to a *demonstration* of the angle in a semi-circle property being suitable, as a response to 'devising an explanation'; some of the students did give dynamic visual representations of the property.

On Joanne's answers

Joanne makes no use of any property of a circle; in other words she communicates no sense of the 'formal structure' that the concept of 'circle' entails. A circle is something she can see, and she is aware (p J1) of practical limitations on empirical readings: she observes that no drawing will be fully accurate and that she has rounded off a measurement to 1 decimal place. However, she leans on the authoritative acceptance

of the 'fact' of the proposition rather than persevere with a true empirical approach - for she does not find the average of her results, but rounds them down to what the answer 'should' be. On page J3, she symbolises the procedure of her calculations, i.e. she attempts to set up a generalised use of the scalar product, but without use of properties, no result yields. The 'alternative proof', on page J4, is essentially the same as the vector proof - both essentially boil down to the cosine rule, with $\cos(\text{quarter turn}) = 0$, but J does not seem to see this sameness.

This work of Joanne's suggests that she has some inkling of a procedural warrant (how to calculate a scalar product and that zero scalar product implies a right angle as desired) but otherwise the justifications suggested are empirical and authoritative - not mathematical warrants.

On Emma's answers

Emma works with symbols, rather than measurements; she recognises the requirement of generality as 'working with any..' (evidence on all three E sheets). But, as her first attempt indicates, she was not in control of the procedure at this level of symbolic representation: she drew a diagram of what looks to represent a general case then proceeded with the calculations for a very special case; she, initially, perceived the generality as being carried by the variable r . However, Emma redeems herself when on page E2 she attempts a symbolic application of the scalar product which incorporates use of Pythagoras's theorem. While her proof is not in standard notation, it is decipherable and sound. For the part (ii) she has produced a proof of Euclidean structure.

This work of Emma's suggests that, as well as a notion of procedural warrant, she has a sense of deduction and falsification - mathematical warrants.

Epistemic connections - warrants - are not empirical notions: whether a student has a belief justified by a mathematical warrant is not testable per se. For example, Emma's proof on page E3 is acceptable - but it does not imply her epistemic functioning. Warrants are theoretical notions, these examples are presented to indicate what behaviours *might* be consonant with their use.

5.5.4 Belief and proof: Coe and Ruthven's study on students' proof practices

Some recent classroom research by Coe and Ruthven (1994) has also investigated students' justifications of their mathematical results. These students, like those I have just referred to in 5.5.3, were also studying mathematics at the 'higher school' level. The study distinguishes students' conceptions of proof, i.e. in what does a proof consist, with their ideas about the function of proof and of how mathematical insight is achieved. The evidence offered in their article suggests that the students did not have 'proof' as an abstract concept, although they did say that 'proof' was the means by which certainty of mathematical propositions was ensured.

Coe and Ruthven's results can be interpreted in the light of the theory of mathematical warrants for learners that I have been developing: the students may have accepted on authority that proof was the way to be certain about a mathematical result, but few of them used deductive-type proofs within their work. They asserted that proof was warrant, but their beliefs were not thus warranted. The strategies with which they used to present their cases for the mathematical results they had obtained in their investigations "were primarily and predominantly empirical, with a very low incidence of what could be described as deductive." (p 52). The exceptional student in the study did exhibit behaviour that indicates that proof functioned for him as a warrant for belief: "For Rod, however, it is the proof by induction which he has produced that takes him from being 'not that certain' ... to being '100% sure' after the proof" (p51).

Habits of perception and bodily action are crucial to our daily functioning. When we move from an everyday activity to working on an abstract mathematical one, the modes of reasoning are different and the warrants are distinctive. But how obvious is this to a competent, but not exceptional, student? Not very, as the two sets of examples reported above suggest. The current National Curriculum in England and Wales emphasises 'social mathematics'. That is to say, it is a curriculum that is weighted towards statistics and applications of measurement and reinforces a seamlessness between 'the real world' and mathematics. There are advantages to this approach, to be sure; the purpose of mathematics should be clearer for those studying it, and that motivating factor is helpful.

An implication from Coe and Ruthven's work is that any expectation that children would, in the post-Cockcroft era, spontaneously understand the nature and technique of mathematical proof, is refuted. The students had more idea of what the nature of proof was said to be, than propensity to employ proof techniques. This is not very surprising if they have not been taught proof techniques; proofs are difficult! That is why, as stated above in 5.4.4.2, short, visually supported proofs of Euclidean geometry have been considered a suitable introduction to this form of reasoning. It is worth noting that the exceptional student had taught himself 'proof by induction'. Those with, in Kutretskii's phrase 'a mathematical cast of mind' may spontaneously develop the awareness that necessity is not a function of an inductive process, but the appropriately competent 'higher school mathematics' student is likely to need explicit teaching in proof techniques. A caveat is, as the thesis presented in this chapter implies, that having a deductive warrant is more than the ability to regurgitate a proof.

The curriculum designer's challenge, then, is to provide a means of educating students in the distinctiveness of mathematical reasoning without making the enterprise of learning it alienating.

5.6 Conclusion

Formation of doxastic attitudes is the first stage in human knowing about mathematical propositions. The justification of some of these doxastic attitudes defines these attitudes as a person's beliefs. From there, if these beliefs are warranted, they may count as knowledge. In this chapter, I have concentrated, not on the truth or falsity of the proposition that is being investigated, (as we have to do if knowledge claims are being made), but on the various ways beliefs are justified and which of these are warranted.

What warrants are used in mathematics teaching and learning? A classification of justifications which come from a set of beliefs that are typical for beginning school trigonometry were sorted into 'authoritative', 'logical', 'perceptual', and 'procedural'. There is no claim to be exhaustive, and I have already indicated that different warrants might be employed by different people to believe the same proposition. Although 'authoritative' is but one of the forms of justification, and an inevitable one if school

students are to be inducted into a 'community of practice', (in the sense of Lave and Wenger, 1991), I suspect that many kinds of mathematical beliefs held by students are actually only justifiable via this warrant. There is a negative connotation to the word 'only' in the previous sentence because, I assert, that, despite the importance in belief formation, an 'authoritative' justification is not a mathematical warrant. Here is a paradox then: we can't do without a 'community of practice' to support mathematical learning, but the warrants for belief in the mathematical propositions held by that community cannot just be the 'authority' of the community itself; to be mathematical belief, the justification must come from other warrants. In mathematics learning, progress is made when students to shift from using authoritative or perceptual-empirical justifications to using logical or procedural ones. To follow on with the idea of tracking warrants for beliefs: we can envisage the situation where the teacher, having taught the students, might attribute her student's warrant for belief as, say, deductive, yet the student's warrant is authoritative. When it comes to practice, epistemic modality - what is believed, known, or taken to be true - in mathematics learning is significant.

Unless awareness of type of warrant is brought to the fore, students' beliefs about mathematical propositions are likely to remain at the level of 'Do I assent to this or not?', rather than, 'If I am to assent to this proposition, what is its warrant?' So for the student the question to be asked is: what is the sort of justification that has helped, or might help, me believe this proposition, and is this the same sort of justification which will enable me to know the proposition's truth value? For the teacher, the analogous question is, through what justification am I expecting the students to take on this belief, and is this the same form of justification through which I hope that they will warrant it? As I have said, a change in the method of justifying belief in mathematical propositions is both essential to the student's formation of mathematical knowledge and difficult to achieve.

6. Chapter 6: Mathematical Objects

The real problem that confronts mathematics teaching is not that of rigour, but the problem of the development of 'meaning', of the 'existence' of mathematical objects. (René Thom, 1973)

Content is the key. There seems a need now to return to the mathematical objects we are studying (Richard Noss, 1995)

6.1 Introduction

This chapter is concerned with the 'stuff' of mathematics: numbers, geometric properties, ratios, theorems etc.; these are - in some sense - 'mathematical objects'. In the previous chapter, on mathematical beliefs, their genesis and growth, I argued that mathematical knowledge required a 'mathematical warrant', the nature of which was discussed. Here, I shall argue that mathematical knowledge requires an ontological commitment to mathematical objects. This argument, then, firstly requires a discussion of mathematical objects, (including in what sense this term has a referent), from which the question of why 'knowledge' requires ontological commitment is developed.

In this chapter, I argue for - in the technical sense as discussed in chapter 4, - realism in the philosophy of mathematics. This is part II of the overall thesis. This is to develop a referent for 'mathematical objects', 'commitment' to the existence of which is part of learning mathematics. While this advocacy of realism may be inspired by mathematical experience, the motivation to develop an understanding of what this philosophical mathematical realism means and implies is generated from my interest in finding out how these ideas apply in education. My argument does not deny the importance of agreement (within a 'community of practice', say), nor the power of notations, but demonstrates a realist vein in mathematics. I want to claim that mathematical objects are not 'just' notations - as in the nominalist position - or that mathematical objects are not 'just' socially agreed conventions - as in the conceptualist position. And furthermore, that this hybrid position - with a realist core - is educationally apt.

Realism's standard ontological position is that there exist (at least some) mathematical objects. This statement is vigorously disputed by Kitcher who presents a realist theory based on idealised actions rather than objects (see 6.6). For educational considerations, epistemological consequences of this ontological position are obviously important. This is where the notion of 'ontological commitment' comes in.

Let me recap on what I am trying to show: from a teacher's point of view, the students or pupils (in H.E. or school, respectively) need to 'grasp' 'objects of mathematics' to be able to progress in the subject. For example, we do not expect great success of students who try to solve a differential equation without the notion of rate, or try to work out percentages without the notion of a hundred. However, the scare quoted terms entail, respectively, epistemological and ontological assumptions, which are entwined. The assumptions or ideas underpinning the phrase 'grasping objects' could be realist or anti-realist. I argue that a realist position is philosophically defensible, compatible with mathematical practice and educationally efficacious.

6.1.1 Outline of the chapter

I try to make a case for philosophical realism in higher school mathematics as follows:

- (a) some other philosophical positions are criticised briefly
- (b) promising realist theories of mathematics are presented and critiqued and an educational version presented
- (c) the notion of 'ontological commitment' is discussed generally and with reference to mathematics students

Explicitly, the thesis is developed as follows: Firstly, in 6.2, referring to the philosophical schools described in Chapter 4, I make a case for rejecting nominalist and conceptualist ontological positions, (that mathematical entities do not exist, that mathematical entities are linguistic (in a broad sense), respectively). By elimination this would seem to leave a realist position. Then, in 6.3, further weight is given to the realist conception in terms of

Chapter 6: Mathematical objects

evidence from practitioners and a defusing of some of the folk-interpretation of platonism.

I then turn to different contemporary realist theories of mathematics. Maddy's forthright realist theory is presented and critiqued (6.4). then Resnik's and Bigelow's views are presented more briefly in 6.5. In 6.6, I review Kitcher's position which challenges the notion of mathematical object from a realist perspective with historical sensibilitites. I follow this with some more historians' interpretations of abstractions in mathematics. From an amalgam of these, I put forward an 'educational realist's' view in 6.8. I claim that this realist conception can support a non-metaphorical notion of 'ontological commitment', which is the notion that I am claiming is integral to learning mathematics.

In order to make this claim, 'ontological commitment' has to be explained. This I do in 6.9, with reference to British-American philosophical tradition and to classroom experience. I then summarise my claim that a realist and abstract sense of mathematical objects can be defended.

6.2 Towards an argument for realism in discussing mathematics learning

Here I suggest some inadequacies with some theories competing with realism. If these criticisms are accepted, then in a simplistic sense, this leaves realism by default (except that another, further, category could then be conceptualised!) Criticism of anti-realist positions are presented in order to prompt consideration of the desirability of the realist alternative rather than to prove it.

6.2.1 Rejection of nominalism

The nominalist thesis was not designed with genetic epistemology in mind, i.e., problems about coming-to-know did not stimulate this philosophy of mathematics which claimed that, ontologically, mathematics was redundant. If, as nominalists claim, mathematics is but 'useful fiction' for describing science, then what other way would there to be to learn about this system of language and derivations if not through science? That is,

mathematics, learnt authentically, from a nominalist stance, is learnt as a set of notations for science.

So what are some consequences of really taking mathematics as 'just notations' used in science? The nominalist assumption is that there is nothing but the physical phenomenon⁴⁶. The following example from Newtonian mechanics is intended to illustrate the problem of taking the 'formal fiction' line. I think that it reads quite strangely, but I believe that this is because I genuinely find nominalism hard to conceptualise as a philosophy of mathematics which includes epistemology as well as (lack of!) metaphysics.

A stone is dropped from a known height. What mathematics can be learnt from the physics of the situation, given a Newtonian framework?

The point of posing this question is to investigate problems of epistemological access when mechanics is ontologically prior to mathematics, i.e., when a nominalist position is taken.

I shall consider nominalism from two perspectives: scientific priority and formal priority and illustrate why neither perspective is likely to be a suitable one for learning mathematics.

6.2.1.1 Mathematical generalities; scientific particularities

The stone falls. Newton's theory tells us that the stone 'accelerates'; is this not a mathematical concept? Not quite: a 'mathematics' student in the nominalist school would be able to make and record measurements, from which she could be reasonably expected to learn about acceleration. But acceleration is a particular instance of a more general concept of rate of rate of change. How does a student in the nominalist school grasp this generality, which is typical of mathematics? She cannot do it in a 'nominalist' fashion,

⁴⁶But the notion of 'regularity of nature' depends on structure: mathematical notions are already implicit.

Chapter 6: Mathematical objects

because, by definition, only particulars can be apprehended. Nevertheless, the generality of the second derivative does not exist outside of its presentations in science (which is a consequence of the scientific nominalist position). This is paradoxical and, so, unsatisfactory.

6.2.1.2 *Just notations: 'black boxes' and meaning*

The discussion above took nominalism as a system accepting scientific ontological priority. While this is very much Hartry Field's approach, often nominalism is construed as formalism⁴⁷, where a 'meaningless', but logically precise, system of deduction is imported to science. In this conception, the time, say, the stone takes to fall from rest is proportional to the square root of the height from which it was dropped. The formula is a 'black box', it has no meaning in and of itself.

From a classroom point of view, this 'black box' approach is both recognisable and undesirable. It is recognisable because, despite intentions to the contrary, students do (sometimes) 'just' plug in data to their 'black box' of a formula to yield a numerically acceptable answer; they do answer instrumentally rather than relationally (Skemp, 1976). It is undesirable because manipulation of a formal 'fiction' is an anathema to the aim of the students 'grasping' the notion for themselves, for it to be meaningful for them.

So desirable relational understanding is essentially anti-nominalist. That is, relational understanding involves apprehension of structures - such as 'proportional to the square root' - which are not fictional. Such structures are examples of 'mathematical objects'.

6.2.2 *Rejection of conceptualism*

The idea that mathematics can aid our conceptualisation of the world is indisputable. But the conceptualist thesis goes further: it states that there is no essential mathematics;

⁴⁷ Hilbert, doyen of formalism, did not work mathematically this way. According to Courant (1980), Hilbert's own epistemological access to mathematical ideas was through cases study analysis and intuition!

mathematics is (just) part of our conceptualisation of the world. Indeed, mathematics is a profound, detailed, symbolic language; truths of mathematics are a function of the rules and constituents of that language, verified by the community of users⁴⁸. Knowledge of these truths requires, therefore, prior initiation into the community of users. The particular conceptualist thesis due to Dummett, was discussed in chapter 4. In this section, I develop a more specific argument against conceptualism in mathematics education, through analysing a recent paper by the social constructivist mathematics educator Paul Ernest (Ernest, 1997). Social constructivism is a sub-branch of conceptualism in Quine's (1953) and Tiles's (1996), sense which I have adopted⁴⁹. Before turning to Ernest's paper, I make two general criticisms of social constructivism.

Firstly, mathematical knowledge can exist outside the mathematical community although, clearly, communication of mathematical ideas is helped by a common notation and recognition of rules of inference. For example, Ramanujan's notebooks are fiercely difficult to penetrate (e.g., Berndt, 1994), but those able to get to Ramanujan's results, even though he was outside their community, have, nevertheless, found propositions which were true. The existence of this man and his results can be construed as evidence of mathematical knowledge existing outside of the mathematical community. It can be denied as being such evidence if (a) Ramanujan's results 'were not knowledge' or 'were not true' until Hardy checked them (b) Ramanujan was, despite isolation and idiosyncratic education, a member of the mathematical community. I do not accept these objections. An alethic conception of truth does not require a truth be known. The notion of a community does not include all those, hitherto unknown to the community, who may yet do something the community values, for this logically includes everybody! Hence, the

⁴⁸ The idea that 'knowledge exists only on the social plane' has been attributed to Vygotskii (Lerman, 1994b), but I interpret what small proportion of Vygotskii's writing I have read as making the weaker statement to the effect that there is such a thing as social knowledge. And I agree with this, (Van der Veer and Valsiner, 1994, p353).

⁴⁹ Kitcher does not have quite the same sense of conceptualist. His vital cleavage is between a priori and a posteriori, rather than between realism and anti-realism.

possibility exists for knowledge to exist prior to being shared. This is my first objection to conceptualism (in the form of social constructivism).

My second objection comes from the conflation of concepts this perspective seems to involve: in particular, the theory conflates epistemology and metaphysics due to defining away the existence of abstract objects, (although the terms may remain as linguistic items⁵⁰). The paradox that we (cognising beings) can know about abstractions is resolved, but only at the expense of doing away with one of the concepts involved (abstractions)! Conceptualism is a possible approach, to be sure, but one which is not mathematics-practice centred, i.e. it does not seem to help in answering the question which is concerned with learners' 'what is it to know specific items of mathematics, like 'infinity' or 'axiom' or other specific abstractions?'. While conceptualists are able to construe 'knowing mathematics' as part of a social discourse, the crux of the individual learning specific mathematics is avoided.

I now turn to a more detailed analysis of a conceptualist notion of mathematical objects:

6.2.2.1 Outline of Ernest's conception of 'mathematical objects'

Paul Ernest's recent paper on mathematical objects sets out to explain in what sense the "objects of mathematics" exist, what they are like "and indeed their objectivity itself" (Ernest, 1997, p1) from a social constructivist perspective. There is no argument for taking this perspective, for Ernest's position in this regard is explained more fully in his 1991 book⁵¹.

⁵⁰ Dörfler (1996) goes a step further - he assumes that "mathematical objects can't be granted an ontological existence and reality" and then claims that any metaphorical talk about mathematical objects leads to a sort of "psychological Platonism", which is, from his view, a priori unacceptable.

⁵¹ In this 1991 book, Ernest's adopts a social constructivist view of mathematics because, roughly, that philosophy of mathematics is anti-'absolutist', descriptive (rather than prescriptive), accounts for fallibility and mathematicians' practice. (Ernest, 1991, pp18-26)

In this 1997 article, Ernest gives arguments for his explanation of the nature of mathematical objects as "among the social constructs of mathematical discourse" (p1). In particular, he claims that the idea that there is something real "behind the signifiers is the result of the reification which is part of mathematical culture" (p7). This idea can be interpreted on two levels:

- a) "the objects of mathematics ... are cultural constructions", (p7), can mean that π , probability and polyhedra, and so on, are *object-like*, in the sense that they are considered by the culture to be 'enduring and objective'; this is the social statement.
- b) "the objects of mathematics ... are cultural constructions" can also mean that π , probability and polyhedra, etc. are 'enduring and objective' to members of the culture; this is the individual's statement.

I want to try to clarify his two-stage aspect of the cultural dimension through an analogy: say British culture considers some defined group of people⁵², X, worthy of compassion; this is the social statement, but no individual attribution of compassion can be made. Irrespective of culture, an individual may have a feeling of compassion towards members of X. As should be apparent, I believe compassion can exist independently of whether it is nurtured by a culture. And those raised in a community in which compassion is considered a virtue, may yet not experience the feeling. So analogously, the 'enduring and objective' nature of mathematical objects need not be felt by all members of the community and may yet be felt by someone outside.

The method, by which Ernest tries to show that a social constructivist account can explain mathematical knowledge, is semiotic analysis, inspired by Rotman (1988), (p1). Ernest clarifies some technical semiotic terms - signifier/ed; signifier token/type - then investigates the relationship between the signifier, (symbol or notation), as well as the signified, (mathematical object). He asks about the connections between signifier and

⁵²Set X could be AIDS sufferers, ex-servicemen, people with Down's syndrome, etc.

Chapter 6: Mathematical objects

signified. Specifically, he challenges that no guarantee of identity can be given to the signified - this is the main point he wishes to argue.

Ernest's argument proceeds, firstly, by using Popper's 'World 3' (of objective abstract knowledge) to explain a feasible connection between mathematical object and its symbolic referent via the 'objective knowledge' in received mathematical texts. He then aligns 'World 3' with a 'transcendental Platonism' (using Irvine's 1990 term, see below, 6.4.1.4), which is contrary to the social constructivist hypothesis.

The second phase of the argument is to explain the relationship between mathematical objects and texts. For if 'World 3' itself is not acceptable, there is a lot of seemly 'objective knowledge' lodged in texts. So, in what sense do the content of mathematical texts have 'object status'? Ernest quotes the continental philosophers who advocate that there many interpretations of texts, "texts do not have unique signifieds" (p6). This potential indeterminacy is corrected by Ernest's Wittgensteinian solution of a "holist conception of meaning" (p6): the participants in the language game create the meaning of the textual item. This meaning is not unique, but mediated by contextual use and normative discourse. This comes to Ernest's main conclusion: "the ontology of mathematics is given by the discursive realm of mathematics" (p7).

6.2.2.2 *An argument against this conception of mathematical objects*

My first objection to this thoroughly thought-through view of mathematical objects is, despite all the scholarly detail, that the proposition that is either trivially true or not proven.

Under what circumstance is this proposition 'trivially' true? Recall, that from the social constructivist perspective "objectivity itself will be understood to be social" (Ernest, 1991, p42). That is, 'objectivity' is defined in such a way that social knowledge 'must be true'. Then if mathematical discourse is an aspect of socially received knowledge, its 'ontology', i.e. its objects, may have 'objective' properties. So the proposition is trivially true because Ernest accounts for the 'objectivity' of mathematics by construing the

Chapter 6: Mathematical objects

discourse of mathematics, part of the agreed knowledge of the community of mathematicians which includes the 'true by social fiat'.

Under what circumstances is the proposition "the ontology of mathematics is given by the discursive realm of mathematics" not proven? To try to show this, recall the more standard English meaning of the word 'objective', (for to assume Ernest's definition reduces the proposition to the trivial case discussed above): "Belonging not to the consciousness or the perceiving or thinking subject but to what is presented to this, external to the mind, real", (Concise Oxford Dictionary). Now, I claim that there are two notions of 'discovery', for which we have evidence, which are themselves evidence that there is more to mathematical ontology than discourse.

The first is the notion of 'unwelcome discovery': 18th century algebra techniques, when applied to infinite series, gave false answers, or paradoxes. The mathematicians of the day, Cauchy et al. were disturbed by "unreliable tools" (Kitcher, 1984, p 249).

The second is the notion of discovery of a new particular: to show this I quote W. W. Sawyer in his reply to Ernest and others (Sawyer, 1992). Sawyer explains that he has a conjecture about the eigenvalues of a certain integral equation. He says:

"But I am sure that the guess is concerned with something of an objective nature. It may be true or it may be false. Whichever it is, once the problem has been formulated with this degree of precision, the matter is out of our hands. There is nothing anyone can do to alter the answer." (p46)

Ernest has not convinced me that the 'objectivity' of unwelcome discovery or of 'new particulars', of which Sawyer writes, are integral to the 'discursive realm' of mathematics.

While I do think the substance of what Ernest is saying does not convince me, there is much in the spirit of what he is trying to get to which is vital and relevant to thinking about 'how to come to know' mathematics. For example, in the 1997 paper, he does try to unpick the detail of the nature of the entities of mathematics with respect to the media ('texts') through which they are communicated. By using semiotic theory, Ernest develops

an account of the relationship between notations and their referents. Signifiers are to be taken as identical when there are "permitted transformations" (p5). But, true to form, these transformations which convert one signifier into an equivalent one are deemed to be part of the language game or 'form of mathematical life'.

6.2.3 Towards realism: summary

From the three basic ontological positions, nominalism, conceptualism and realism, I reject nominalism and conceptualism. The problem with nominalism is that it is impossible to learn *mathematics*, all we can get to are formal structures and conventional notations. Teaching experience and cognitive psychology explain the paucity of that approach. The problem with conceptualism is that it fails to link to the material world of physical experience, save by linguistic conventionalism which has no necessary force. In particular, a strict interpretation of a 'social constructivist' perspective à la Ernest, is either trivial (what a 'community' claims to know is knowledge because knowledge is determined by the 'community') or false (mathematics can theoretically be discovered by those outside a 'community'). This rejection of formal ontologies does not mean that I do not recognise nominalistic or conceptual aspects inherent in the enterprise of learning and teaching mathematics. Perhaps it is because so much in current British school education is formal and conventional that I feel motivated to try to develop a position which expresses 'what mathematics really is' outside of marks on paper or social discourses.

6.3 From Plato's beginnings ...

Plato invented a wonderful image: the world of Forms. It has been such a powerful metaphor that it is still in Western culture after more than two thousand years! In terms of the discussion here, what is the seed that Plato's dialogues contributed to contemporary ideas about mathematics and education? I claim that, at least, Plato offered the idea that mathematics is stable, that mathematical truths are not a function of time or place or person. From this we get the idea of mathematical truth as independent of given minds - mathematical propositions are true by virtue of what *is*, albeit in the world of Forms, the epistemological accessibility to which requires argument. This conception of

Chapter 6: Mathematical objects

mathematics is fertile ground for the notion of an 'objective' mathematical entity. While post-Kuhnian historians of mathematics have good arguments to challenge this view (Gillies, 1992, Grabiner, 1986), at a naive level, ideas like 'the Circle Form' can make sense as a way of understanding circles conceptually. Even if this naive sense were granted, as it is well known, where Plato's ontology becomes unacceptable is in its link with his epistemology: Plato's theory is that we 'recollect' knowledge from a previous life (e.g. Meno, trans. Guthrie, 1956, pp 138-9). Although this solves the inaccessibility problem, it is untenable given today's cognitive science, where causes for cognition are sought in our current life-time!

6.3.1 *On accounts of mathematicians' practice*

'Platonism', roughly, the idea that (some of) the content of mathematics as permanent and objective, is still alive in mathematicians as well as philosophers. In a popular book, the mathematician Roger Penrose, without obvious irony, claims:

"One's mind makes contact with Plato's world whenever it contemplates a mathematical truth, perceiving it by the exercise of mathematical reasoning or insight." (1989, p 205).

Another mathematician, makes a similar claim more defensively:

"I must warn you that, as with many mathematicians, I am at heart a Platonist. I don't know whether we invent or discover mathematics, but I do believe that, at least once a mathematical object has been invented/discovered, it is independent of us: it is objective." (Gold, 1994, p 21)

And from Davis and Hersh there is the humorous adage, well known in mathematics education circles, that:

"the typical working mathematician is a Platonist on weekdays and a formalist on Sundays" (1981, p 321)

Chapter 6: Mathematical objects

Not all mathematicians need have platonistic tendencies. Poincaré, as quoted by Hadamard (1945) was an idealist. But it seems to be a psychological attribute of practice that, at least at the moment of mathematical work, the mathematical content has a referent as real as love or death or a handful of radishes. The philosopher's job is to make sense of such data.

6.3.2 *Is platonism the same as realism?*

Certainly these terms are very close in meaning and they are used synonymously by some (for example, Davis and Hersh, 1981). But there are some distinctions. In my reading, it is possible to be a platonist and an idealist - an exemplar of this position is that of Berkeley. The position of Field, or other similar nominalists, is that they are realists, with respect to science, but deny mathematical entities at all. Hence they cannot be platonists. But this argument is open to the criticism that they are not *mathematical* realists at all. Be that as it may, my point here is draw attention to some subtleties in these terms.

To show that platonism does not imply realism: Idealism is a term, older than Dummett's 'anti-realism', that connotes the position complementary to realism. Berkeley was an idealist in the sense that he did not attribute existence, per se, to objects of experience⁵³. He was also an empiricist because he took sense-data as the primary cause of knowledge. But these data were not caused by the objects in themselves but by the mind of God.. Berkeley was a platonist for he would have asserted that mathematical entities do exist independently of human thought.

6.3.3 *Platonism and realism: summary*

The key thing about realism in mathematics is that there are mathematical propositions, the truth-value of which are not determined by human consensus, (even though, trivially,

⁵³ Although Kant considers Berkeley an idealist - and argues against his position - (trans. Kemp-Smith p 244), Dummett suggests that Berkeley was a "sophisticated realist" (Dummett 1992 p 464). Dummett takes this view, I suggest, because Berkeley's notion of truth was not semantic.

human agreement about such a truth value is a function of a consensus). Platonism, by contrast, does not have such a tidy definition. For some writers it is a term connoting the advocacy of Plato's Forms - either disparagingly like Davis and Hersh or acceptingly like Penrose - for others it is a convenient label to indicate objective existence in mathematics. And it is this perspective which contemporary philosophical writers tend to employ. For example, adapted from Irvine's list of attributes of platonism, Irvine, 1990, p xix, discussed further in 6.4.1.4., we have the stipulation that mathematical entities are not space-time particulars, yet we can refer to them and have knowledge of them, and the statements of mathematics possess truth-values independent of human thought.

In the following sections, 6.4 - 6.6, I present some contemporary realist theories from the philosophy of mathematics, which will be interpreted with respect to their efficacy in an interpretation for mathematics in education. The theories are of two main types distinguished by whether their primary focus is ontology or epistemology. As a representative of the 'ontological' theories, I focus on that of Maddy (principally from her 1990 book), and briefly mention Resnik's and Bigelow's positions. As a representative of, 'epistemological' theories I focus on Kitcher's (from his 1984 book). In either case, ontology requires explanation of how we know and epistemology requires explanation of what we know. In terms of application to education, Maddy's ontological precision is nice but difficult to interpret for learners, whereas in Kitcher's wide ranging evolutionary realism it is difficult to specify what students learn experientially, despite the reality he attributes to mathematics.

6.4 Maddy's theory

Mathematics is part of the physical world - parts of it are perceptible! This is Maddy's claim which she has been developing since the early 1980's. It is a view distinct from, though not incompatible with, the notion that mathematics is part of a social world. Maddy calls her distinctive philosophy of mathematics 'set theoretic realism'. In a nutshell, the theory is based on the claim that sets can be perceived. The purpose of this is to allow the modern foundation of mathematics - set theory - to be realised perceptually. This gives a perceptual core to these mathematical foundations and hence the crucial

Chapter 6: Mathematical objects

scientific link she wants for her explanation of mathematics which entwines science with mathematical practice.

Maddy's theory links mathematicians' practice with the physical perceptible world. Links between these two domains are, of course, of interest in mathematics education.

6.4.1 *Background to Maddy's realism*

The influences shaping Maddy's developing theory are briefly described in the next four sections.

6.4.1.1 *"Quine/Putnamism"*

Quine's influence on Maddy comes chiefly through his theories of naturalised epistemology. This is the idea, briefly, that we cannot stand outside our theories of the world:

"The old epistemology aspired to contain, in a sense, natural science; it would construct it from sense data. Epistemology, in its new setting, conversely, is contained in natural science as a chapter of psychology. ... There is reciprocal containment...: epistemology in natural science and natural science in epistemology" (Quine, 1969 p 83)

Maddy reports that Quine rejects the double standard of considering mathematical entities as purely linguistic, but physical entities as real, and that this is compatible with Putnam's view that we could not have contemporary physics even formulated without mathematics. Thus these philosophers endorse an 'indispensability thesis': mathematics is indispensable for science (e.g. Maddy 1989, pp 1132-1133). In other words "successful applications justify, in a general way, the practice of mathematics" (Maddy 1990, p 34). So, if it is granted that advanced scientific theories have an integral mathematical component, (quantum mechanics is a standard example), then if the science holds, the mathematics is also confirmed. "Quine/Putnamism" is the term Maddy coins for their judicious amalgam of 'common-sense' and 'scientific' realism: what common-sensically

exists are those medium-sized objects of our perceptual experience and “the considered judgement of science is the best justification we can have.” (1990, p 13). This is, thus, a ‘two-tier’ theory without a particularly mathematical component: the lower tier is common-sense realism the higher tier is scientific realism in the form of current science.

6.4.1.2 Gödel’s platonism

Gödel also advocated a two-tier philosophy of mathematics which was similar to Quine-Putnam’s on the higher level. Unlike Quine and Putnam, Gödel’s ‘lower tier’ was tied to mathematics. This lower tier consisted of intuitions of mathematical truths. These necessary truths were perceived by a faculty analogous to that of sense perception in science. As Maddy puts it: “the simpler concepts and axioms are justified intrinsically by their intuitiveness; more theoretical hypotheses are justified extrinsically, by their consequences. This second tier leads to departures from traditional Platonism similar to Quine/Putnam’s. Extrinsically justified hypotheses are not certain, and, given that Gödel allows for justification by fruitfulness in physics as well as mathematics, they are not a priori either.” (1990, p33)

Gödel was significant to Maddy (Maddy 1990b, p 266). For after all, she wanted to explain the ontology involved in mathematical practice and here was a mathematician, with impeccable credentials, who advocated a form of realism. Gödel’s philosophy contained the “flabby” (p 35) notion of intuition at its lower tier but concurred with Quine/Putnamism at the higher tier. How can Gödel’s theory be made philosophically rigorous? Maddy’s ‘set theoretic realism’ is an attempt to keep the spirit of Gödel’s philosophy within a stricter analytic philosophic framework. She aims to replace the lower tier (of either theory) by set theory perceived. In this way she should satisfy both Gödel and Quine/Putnam: the lower tier is both mathematics and it is science.

6.4.1.3 Realism and epistemology: the problem of access and the Benacerraf syllogism

Without begging the question about the existence or otherwise of mathematical objects, there is a well received sense in which mathematics is supposed to be about abstractions.

Roughly, it means that what one works with in mathematics are not space-time particulars, for example, the motion of a ball through the air may be modelled by a parabola (given requisite parameters). The particular motion of the ball is indeed a space-time particular, but the structure of the model is a mathematical abstraction. The members of the class of such abstractions are ‘mathematical objects’.

The classic question, going back to Plato, is ‘how do we get to know about these abstractions if they are not part of our experienced world?’ For although the path of the ball is part of the physical world, the parabola-as-model is not. As I have said, Plato’s answer was that we recollect them from a prior existence. Anyway, theories of knowledge evolve in time and in the 1970’s a theory known as the causal theory of knowledge (see Benacerraf, 1973) was in vogue with analytic philosophers, the school of whom nurtured Maddy. This theory asserts that

(i) *there must be a causal connection between that which makes a belief true and the epistemic subject holding the belief.*

Now, think back to these ‘mathematical objects’. They are abstract. This means that

(ii) *they are causally inert,*

they do not change, like the energy of a moving particle on interaction, nor do they stimulate the senses as my cup of coffee does. Benacerraf (op. cit.) noticed that (i) and (ii) implied that there is no possible knowledge of mathematical objects! Although Maddy observes, in her 1984 paper, that the causal theory of knowledge has been refined to a ‘reliabilist’ theory, this ‘Benacerrafian syllogism’ captures quite nicely the traditional and pervasive problem of gaining access to the subject matter of ‘ideal’ subject matter typical of mathematics.

6.4.1.4 Physicalistic platonism

In 1990, a collection of articles, ‘Physicalism in Mathematics’, edited by A. D. Irvine, was published as a result of the *Physicalism in mathematics: Recent work in the*

philosophy of mathematics conference, held at the University of Toronto. Maddy's article in this collection is entitled 'Physicalistic Platonism' (Maddy, 1990b). This title is a provocative oxymoron which indicates her determination to work with truly mathematical entities within science! Maddy chooses to develop her theory under the 'platonistic' label even though her theory does not concur with all of the features of platonism listed by Irvine in his introduction to the collection (Irvine, 1990, p xix). In particular, Maddy rejects no. (ii) on Irvine's list of points characterising platonism: "[mathematical] entities are non-physical, existing outside space and time". She rejects this notion of an inert abstract entity (Maddy, 1990, p 21) because of the 'accessibility' issue discussed above. In Irvine's terms, Maddy's theory appears to be that of an "immanent mathematical real[ist]" (Irvine op. cit. p xx) for she is a realist who wishes to construe mathematical entities naturalistically (i.e., as part of the world). Irvine categorises Maddy's position as 'physicalistic', but not that of a 'transcendent' (i.e., platonic) realist. In contrast to some versions of platonism, Maddy demands neither the necessity nor a priori status usually associated with the platonistic label. In short, Maddy develops a theory that sees mathematics as objective *and* fallible, involving 'entities' which are *not* outside our experienced world (as Plato's original Forms were and Irvine's condition (ii) retains). Nevertheless, Maddy's motivation to find a physical germ to mathematics certainly locates her in the 'physicalist' camp. Physicalism is described by the nominalist philosopher of mathematics Field as:

"the doctrine that chemical facts, biological facts, psychological facts, are all explicable (in principle) in terms of physical facts. The doctrine of physicalism functions as a high-level empirical hypothesis, a hypothesis that no small number of experiments can force us to give up. It functions, in other words, in much the same way as the doctrine of mechanism (that all facts are explicable in terms of mechanical facts) once functioned ... Mechanism has been empirically refuted; its heir is physicalism." (Field, 1972, quoted by Maddy 1990b).

This idea is essentially that of the 'ontological unity' of Steven Rose, (Rose 1997). Whether 'physicalistic platonism' does make conceptual sense or not, this much is clear,

Chapter 6: Mathematical objects

Maddy is advocating 'real' mathematics in the real world. I shall now turn to some of the basic detail of the theory:

6.4.2 *Back to realism à la Maddy: Set theoretic realism*

Maddy's aim is to find mathematical entities that are directly perceived and that are foundational with respect to mathematics. This will enable her to replace the 'intuition' of Gödel's platonistic theory with a theory that is both philosophically respectable (it does not rely on 'intuition') and mathematically respectable (its foundations are themselves mathematical). Her candidates for such entities are sets, objects of set theory. If she can show that some sets are directly perceived, then she can link these objects of basic perception with the sophisticated and abstract mathematical notions that constitute higher mathematics. It is a clever move: set theory underpins modern mathematics. Functions, for example, are defined set theoretically, this implies that, say spaces of functions - an abstraction from function - also have a set theoretic description. This is already well established mathematics.

The claim is that we can perceive (at least some) mathematical objects *qua* mathematical entities and not essentially the abstraction of a scientific (say, physical) notion. For example, a pile of stones is a physical object, on a human scale - rather than on a chemical or geological scale - and the pile can be considered independent of specific cultures, where it might be a sculpture or cairn. Maddy can perceive it as a set which, being a set, therefore has a number property. So, it would seem that this set theoretic *and* perceptual foundation could satisfy the claim that there can be non-metaphorical mathematical objects *and* they can be known in an analogous way to some physical objects.

6.4.2.1 *How to perceive a set*

If perception of a set is a physical phenomenon, it should have a physical explanation. Maddy relies on the neurophysiological theory of Hebb (1949) to justify her claim that "we can and we do perceive sets, and that our ability to do so develops in much the same

way as our ability to see physical objects." (Maddy 1990, p 58). Hebb's particular theory is not itself of importance here. What is important for the validity of Maddy's theory, from her naturalised epistemological framework, is that there does exist (or, even, could exist) a scientific explanation for how a set might become an object of perception.

We learn to perceive sets by repeated exposure to them. Eventually, when opening an egg box, we perceive the set of three chicks, or whatever, without a count process. The set with number property three is perceived directly; it does not matter what the presentation of the set of three objects is, we can 'see past' the particular trio to the generality - the 'three-ness' - of the triple.

Maddy wants to assert the perceptibility of something mathematical so that the status of the truth value of mathematical propositions is shot through with the surety of physical objects. This will allow a replacement of Gödel's 'lower tier' of intuition with a scientifically based explanation of how these mathematical objects are linked to a physical world.

There are, clearly, all sorts of objections this theory must face. Before bombarding it with these, I want to draw out some points that make it attractive:

- 1 - it has been designed to be foundational in a physical and mathematical sense: to this end, it is simple, imaginative and bold
- 2 - unless you decree that nothing mathematical can be perceived, it is difficult, given the concept of set, to outright deny the claim that a set has been perceived thus
- 3 - it provides a theory of transition, from the physical, experienced world to a mathematical world underpinned by set theory

It is this last point, 3, that is particularly interesting to me. This is because I see making the transition from, loosely, engaging in physical activity to understanding abstract mathematics as a central one in learning mathematics. A theory, such as Maddy's, allows, in Rose' terms, ontological unity while making this epistemological transition.

6.4.2.2 *Some objections to set theoretic realism*

6.4.2.2.1 *Sets and classes and numbers*

To have confidence in this theory, the central issue of set perceptibility must be scrutinised. The challenge that I shall discuss is whether it is a mathematical set that is perceived.

Maddy relies on our naïve perception of 'medium-sized' objects. This granted, she notes, as Frege did, that there is no unique number associated with 'that apple on the desk', so we do not perceive numbers directly. After all, that apple consists of many molecules, (which we cannot directly perceive), and it also has several pips, many colours on its skin and one stalk (which some can perceive). Therefore, material objects are not directly instantiations of numbers. Rather, these material objects of our perception can be perceived in terms of sets and those sets have number properties.

That sets have number properties is clear; what is not so clear is that mathematical sets themselves are perceptible. The word 'set' is in common parlance as well as being a mathematical term. I am not convinced that Maddy does not conflate these two. For example, she explores set-perceptibility through an example of Steve perceiving a set of eggs (whether there are enough for a recipe). Surely this is an ordinary language use of 'set'? How can Steve decide whether he has perceived the mathematical object? I suggest that Maddy has switched language games from that of the kitchen to that of the theorist and back again - not that the boundary between the domain of these two discourses is anything but fuzzy. This objection can be defeated if Maddy can convince the objector that the mathematical notion of set is not as she puts it, "... a linguistic achievement" (p 64).

In favour of an a-linguistic notion of set, recent work on babies (Wynn, 1992) suggests that they can distinguish between sets of small numbers of identical dots. This lends weight to Maddy's assertion that the primitive perceptual understanding related to number is pre-linguistic. But it still does not imply that it is a set that has been perceived.

Milne, in his 1994 review of four different physicalist theses in the philosophy of mathematics, denies the possibility of naïve perception of sets. He claims “Thinking in terms of sets is something we learn ... the mathematical notion of a set [is] a concept that will not arise pre-linguistically” (p310). The reason comes down to the distinction between sets and classes. Milne does allow Maddy's notion of set perceptibility, although without much enthusiasm: “We see the members of sets (sometimes). We may even see sets, for, as I have said, there seems to be no knockdown argument in favour of their being abstract [and so not perceptible].” (p 311) But his more serious objection is his argument that it is not a mathematical set that has been perceived but a classification by properties:

"In Maddy's usage sets are determined extensionally by their members: this is the mathematical notion of collection. Classes, on the other hand, are determined intensionally by properties: this is the logical notion of a collection" (p310)

From a very young age we classify things by their type; this, Milne claims, is the pre-linguistic concept, not seeing the set-hood of collections. To exemplify: a child looks in a field in which there are three ponies and many sheep. That child might classify the two kinds of animal from the perception of sheep-type and pony-type. In Milne's terms these are, for the child, collections, not mathematical sets, even though that child may later learn about sets and perceive set-hood given a similar visual stimulus.

Maddy would argue that, for many subjects, the set with number property three is perceived on looking at those ponies. It is an “impure set”, (Maddy, op. cit, p156), but a set nonetheless. As for the sheep, they are too numerous to have an at-a-glance number property, but if they form a set, there is an associated number property. But this implies that we can perceive sets without perceiving their number properties. This veils numbers from direct perception again; properties of these sets can only be inferred. This objection does not worry Maddy, she is only aiming for some perceptibility at the lowest tier. If this is granted, then the mathematical machinery of set theory guarantees the security of other number properties.

Chapter 6: Mathematical objects

On another, related, tack, we can ask 'How many sets are perceived when I look at a lone apple on my desk?' This question raises the concern, which I shall only mention, that there is an abundance of these potentially perceptible sets. Maddy acknowledges Chihara's charge that the set theoretic realist perceives not only the apple on the desk, but the set consisting of the apple, and the set consisting of the set of the apple, ad infinitum, (Maddy, 1990 p150-2). Her response is to offer these options: *either* 'one can't perceive the difference between a singleton and its unit set' *or* 'deny there is any difference between a set and its singleton' (ibid. p 152). My response to Chihara's objection is that this 'abundance' is a positive advantage to set theoretic realism! For it permits the infinity of mathematical objects from the finitude of physical ones.

Can the question of whether sets are perceptible be an empirical question? I think not. I am quite persuaded that infants and animals may be able to make number distinctions without language, but I cannot infer from that-as-a-fact that it is a set which they perceive. Nor can I infer that it is not a set. There are Hebb-type neurological connections in the brain for all sorts of things that we recognise, mathematical or otherwise. This comes down to the nub of realism: are notions like sets 'just' linguistic or are they part of the intrinsic structure of the discreteness humans and other animals can perceive in the world? Maddy's theory develops from the latter; Dummett's from the former.

6.4.2.2.2 *Triangles, transformations and other mathematical objects*

Different branches of mathematics have their own character. By this I mean there are types of question, ways of thinking and modes of justifying that are different across the sub-disciplines of mathematics. This is a feature of mathematical practice. In particular, set theory has its own character which is different from that, say, of analysis. Although set theory under-pins analysis, the set theoretic definition of 'function', for example in Halmos (1960, p 30), gives little clue to its nature. The way function theorists think about functions is, in practice, different from the way graph theorists do. Maddy acknowledges this notion of mathematical character :

"Even though the objects of, say algebra are ultimately sets, set theory does not call attention to their algebraic properties, nor are its methods suitable for approaching algebraic concerns." (ibid. p5)

but, thereafter, does not attend to these aspects of mathematical practice that involve different ways (set theoretic, algebraic etc.) of mathematical knowing. Maddy would not consider this divergence of conceptualisation a problem, because, as it is mathematics, each socio-semantic conception (of function) can be traced, in theory, back to a set theoretic definition. This common-core gains its reality from the perception of elementary 'impure sets' and the efficacy of mathematical practice in scientific achievement. I would accept this, but Maddy does claim to "develop and defend" (1990, p3) a type of mathematical realism that concurs with the naïve philosophical sense that working mathematicians are supposed to hold, and I do not think her theory does do this because of this notion of mathematical character that she recognises but does not develop.

6.4.2.2.3 *Questions about 'converging ontologies'.*

Maddy, nevertheless, continues to consider mathematical - e.g., geometrical - objects simultaneously as set theoretic constructions and as meaningful entities within a geometric context. But is not a triangle's *reality* not just a function of its set theoretic description, but of its perceptual impact and functional role? Indeed, it is difficult to be sure what she would consider to be a satisfactory set theoretic description of a triangle and how this lengthy precise description would then relate to triangles as used by mathematicians (including learners of mathematics). Maddy's response to the request to give a set theoretic account of a triangle is likely to be: it can be done in theory⁵⁴. So yet again the practitioner's experiential understanding would be superseded by a non-practitioner's theory.

⁵⁴ But recall the effort Russell and Whitehead required to establish $1 + 1 = 2$!

Chapter 6: Mathematical objects

The ontological relationship between sets and mathematical entities is only developed in the case of numbers. Clearly, exemplification of Maddy's theory within every part of mathematics would not be manageable. Nevertheless, the mathematical fields of, for example, geometry, probability and algebra employ notions that have 'entity' status; for example, knowledge of 'random numbers' involves a conceptual base that goes beyond that of any set theoretic definition⁵⁵.

I believe that Maddy would argue that the crucial issue is truth values of mathematical propositions. And these propositions can be reduced - albeit laboriously - to their set theoretic equivalencies. Even if this is acceptable, the more crucial point, for the validity of her account, is to say whether, why and how the geometrical object 'triangle' *is* the set theoretic object 'triangle' for the purposes of mathematical practice. Maddy does answer the 'whether' and 'why': 'yes it is the same because set theory is a foundation for mathematics' is not only her response but also her rationale. The question of 'how' these ontologies converge is not solved.

Maddy's theory gives a flavour of a contemporary realist philosophy of mathematics. In particular, it is a physicalist theory; one that starts from the assumption that mathematics is integral to the physical world, of which human culture forms but a tiny part. In the next section I shall give a brief flavour of two other realist philosophers of mathematics who assert some variation on the platonist theme.

6.5 *Other realist philosophers*

There are several contemporary philosophers of mathematics who consider themselves broadly realist, rather than conceptualist or nominalist. These include Steiner (1975), Shapiro, (e.g. 1993), Azzouni (1994) as well as Resnik and Bigelow discussed briefly below.

⁵⁵ Milne (op. cit.) discusses further the particular problems inherent in set theoretic realism as it pertains to probability theory.

6.5.1 *Resnik*

Another realist conception of mathematics is given by Resnik, (e.g., Resnik 1993). His view, like Maddy's relies on a 'naturalised epistemology'. He considers "mathematical objects [as] positions in patterns, and mathematical knowledge is knowledge about patterns." (op. cit. p 51). Unlike Maddy, Resnik "[does not] think that [any] mathematical knowledge is acquired by something akin to perceiving mathematical objects." (op. cit., p 50). Resnik wants to be able to make epistemological sense out of the ancient notion of Platonist ontology. He does this by giving a "postulational account of mathematical knowledge" (ibid. p 40). His postulational account rests on the way mathematics is used and relied upon in natural science. From the point of view of 'epistemology naturalised', the reality of physical objects is allowed as that is the best theory we have of how the world works. Modern physics, in particular quantum mechanics, relies on sophisticated mathematics and Resnik challenges the assumption that there is "a clear and sharp, causally or spatio-temporally grounded, ontic division ... between mathematical and physical objects" (ibid. p 43). He illustrates this with reference to quantum mechanical particles which, having no definite spatio-temporal location, "seem more like mathematical objects than like everyday, common sense bodies" (ibid. p 46); if we allow the reality of physics, then we must allow the reality of mathematics.

Mathematics is, nevertheless, distinct from natural science. In Resnik's theory, mathematical objects are not subject to change in the same way as physical objects are, although he grants that this could be problematic as a distinguishing feature (ibid. p 45). By defining: "mathematical objects [to be] positions in patterns, and mathematical knowledge is knowledge about patterns." (ibid. p 51), Resnik asserts their abstract unchangingness.

Briefly, I want to discuss some particular points about Resnik's theory:

- (1) The concept of pre-mathematical knowledge
- (2) The notion of paradigm physical objects

(3) The truth of propositions involving postulated entities

(1) Resnik asserts that, because of their “gigantic collection of techniques” (ibid. p51) the research mathematician learns about patterns in a fashion distinct from a learner or a human from the distant past. I am not sure that this is true and he offers no evidence from psychology or cognitive science to support his claim. He rejects attempting to explain the genesis of mathematical knowledge in children as they “have help from those already in the know” (ibid. p 51), although it is not clear what theory of teaching he is assuming here.

Resnik’s explanation of the genesis of mathematical knowledge is a story designed to substantiate his conception of mathematical objects as ‘positions in patterns’. Using his theoretical notion of a ‘template’, Resnik claims that, without yet arriving at abstract objects, ancient peoples could/would/might have “a representational system for designing and thence to playful and creative attempts to explore possibilities” (ibid. p 53). This leads to abstractions ‘in the limit’ for “we will be forced to posit entities, such as points and lines and circles as existing in their own right” (ibid. p 55) For example, the abstract notion of a point appears as the limit of the possibility of cutting a line smaller and smaller and smaller and...Resnik fudges the result of a theoretically indefinite sequence of *actions* with that of an *object* that only exists in the limit and so rests the burden of the existence of the point, say, with this extensive sequence of operations that results in the point. Pre-mathematical knowledge, then, is knowledge without the crucial notion of infinity. Knowledge of infinity is first understood as the possibility of indefinitely executing some operation on a template.

(2) Resnik’s idea that electrons are ‘paradigm physical objects’ is quite different from Maddy, for whom they are the medium sized physical objects of our perceptual capacity. In either case, because both Maddy and Resnik work within the ‘naturalised epistemology’ framework, they both require a fundamental physical object. ‘Naturalised epistemology’ declares that science is to be considered basic, but does not decree what in science is basic. I suggest that what each of Maddy and Resnik pose as fundamental leads each to beg their own question, i.e. to presume their own conclusion. For Maddy’s theory

rests on our perceptual ability, particularly that of sight, and these 'medium sized objects', are objects, that is they exist, if and only if we can perceive them (through sight principally). Analogously, Resnik's claim that no line can be drawn between physical and mathematical entities is bound to be true if what he considers paradigmatically physical is an entity that requires mathematical nous to comprehend it.

Is every ontological search circular? In the paragraph above, I indicate that Maddy and Resnik seem to be circular as they rely on what they are in fact trying to show is fundamental. Kuhn (1962) defines 'paradigms' as achievements that satisfy the two conditions of (i) "attract[ing] an enduring group of adherents away from competing modes of scientific activity [and] (ii) being sufficiently open ended to leave all sorts of problems for the redefined group of practitioners to resolve" (page 10). In this sense, both Maddy's and Resnik's paradigmatic physical objects are part of different notions of the physical world and can be construed as fundamental to those conceptions. Kitcher, too, can be seen as circular as his reliance on 'warrant' is both crucial and unanalysed: warrants are what the community passes on and are warranted because they are what the community passes on (perhaps this is better construed as regressive; still it is unsatisfactory).

(3) In a rather disappointing concluding section to his 1993 essay, Resnik attempts to delineate the difference between different 'positings': "People have posited ghosts, the Ether and phlogiston with as much ease as they have posited numbers. How can positing lead to knowledge in one case and not in the others? What distinguishes between them? Primarily, truth and existence." (op. cit. p 57). His entire theory rests on his own story about our ancestors' construal of the outcome of the limit of possibly indefinitely repeated actions (cutting a line to get an extensionless point). A serious justification of the truth of propositions involving positings will have to go back to the processes from which positings emerge as abstract objects. In some sense these processes are what minds can do to patterns (rather than what patterns can do to minds: as Resnik intimates patterns can impose themselves on pigeons). Resnik asserts that the human mind's creative capacity

Chapter 6: Mathematical objects

allows playful and imaginative manipulation of patterns from which the real, abstract, existent mathematical objects emerge.

6.5.1.1 *Pedagogical implications*

A pedagogical application of Resnik's theoretical perspective on the existence of mathematical objects is the celebration of the 'playful and imaginative'. For it is through the imaginative capacity enjoyed by so many human thinkers that the positings, intrinsic to Resnik's theory, are realised. The other important issue for teaching mathematics that is implicit is the central notion of a limit. I want to now look at these two aspects of the theory with regard to teaching negative numbers.

(a) "Minus numbers go backwards" confidently asserted an eight year old, never having been taught about these 'objects'. Here is an example of an imaginative extension of the forward progression of the counting numbers. The mental play relies upon images of something like the number line and of an enactive, (Davis 1995), counting-on by travelling forward along this 'line'. The context of the child's utterance has been lost, but his idea can be used as a basis of a pedagogical representation of the initial idea of directed numbers. The pattern aspect, that Resnik draws attention to, is evident here: symmetry between the negative and positive numbers helps establish the existence of the former.

(b) Resnik's theory seems to say that we only get genuine mathematical objects when some infinite process has taken place. In which case, the young child who imagined backwards-going numbers (corresponding to the familiar forward-going ones) is only deemed to be talking about a mathematical object in the case that he considered a position on the number line as the result of a limit process. From this point of view, most of what we teach in school is 'pre-mathematics' in Resnik's sense.

6.5.2 *Bigelow*

John Bigelow is an Australian philosopher of mathematics who has developed a view of mathematics, based on "David Armstrong's a posteriori realism" (Bigelow, 1988 p1),

which he claims is both platonist and physicalist. Mathematics is platonist, he claims, because it concerns Forms; it is physicalist because those Forms are instantiated by physical objects, their properties and relations between their properties, (Bigelow 1990, p 291). The notion of 'Form' that Bigelow works with is 'universal' and these universals are physical! Hence these universals' existence can be discovered using a scientific method: "mathematical properties and relations are really there in the world, if only we can manipulate things in such a way as to make them emerge", (1988, p2). To illustrate this claim, Bigelow gives the example of the Pythagorean discovery of mathematical proportions which underlie musical harmonies and their "discovery that natural numbers are not the measure of all things", (1988 p5).

Bigelow's 1988 book tries to avoid epistemological issues. He "address[es] the question of what numbers are, and not how we know about them" (p 4). If I want to apply some of Bigelow's ideas to education, the question of how it is possible to know these number-things is also important. Bigelow acknowledges sympathy with Maddy's and Resnik's approaches to mathematical ontology and to Kitcher's epistemology, to which I now turn, but has a different theory to put forward about the physical nature of mathematical entities.

6.6 Kitcher

Kitcher's work in the philosophy of mathematics is better known to mathematics education than Maddy's (for example, Kitcher is quoted in Ernest 1991). In his 1984 book 'The Nature of mathematics', from which my discussion is (and all page references are) taken, Kitcher puts forward a philosophy of mathematics which uses 'psychologistic' epistemology to develop a theory which he intends to be both Kantian-constructivist and empirical-realist. He also presents detailed work on some history of mathematics which he uses to illustrate the actual development of mathematical knowledge within the academic mathematical community.

Kitcher's aim in this book is to challenge the idea of mathematical knowledge being a priori and to present his alternative 'evolutionary' theory (p92). Kitcher takes it as read

Chapter 6: Mathematical objects

that the standard way of looking at mathematical knowledge is through apriorism. A priori knowledge, by definition going back to Kant, is knowledge independent of experience. And Kitcher rejects the idea that mathematical knowledge is a priori. To do this Kitcher looks at his categories of 'Platonic', 'constructivist' and 'conceptualist' routes to knowledge, divides each into the a priori and a posteriori and argues, in each of these categories, against mathematical knowledge being that category's sort of a priori knowledge. By rejecting every hue of a priori knowledge as pertaining to mathematical knowledge, Kitcher logically concludes mathematical knowledge is a posteriori and proceeds to develop a 'defensible empirical' theory of mathematical knowledge.

Kitcher is concerned with "What mathematics is about? [and] How does mathematical knowledge grow?" (p6). This is different from Maddy's aim, for whom justifying the ontological foundation for mathematical objects was central to her project, despite the fact that that included an epistemological analysis. Of course, mathematical propositions, the knowability of which are Kitcher's major interests, involve mathematical concepts. Kitcher's thesis is that these concepts evolved from a perceptual base through 'rational transitions' in 'mathematical practice'. His notion of mathematical practice is carefully defined; it consists of five components: "a language, a set of accepted statements, a set of accepted reasonings, a set of questions selected as important and a set of mathematical views" (p 163). Types of rational transition are also suggested, corresponding to each of these components (pp 170 - 192). Kitcher relies on a community that passes on and develops mathematical knowledge. "I shall suppose that the knowledge of an individual is grounded in the knowledge of community authorities." (p 5) The realist core of this 'evolutionary theory' is that the "origins of mathematical knowledge ...[are] warranted by sense perception" (p 96). So, the primitive matching of, say, three people with three bowls lead, over time and through teaching, to abstract notions of cardinality.

From an 'experiential stance', such as I take, it would seem that Kitcher's arguments against mathematics being independent of experience would be helpful. Overall, I think they are, even though he labours his rejection of the a priori. However, there are several aspects of Kitcher's theory of mathematical knowledge with which I disagree or find

unhelpful within my task of conceptualising mathematics-in-education knowledge. For example, while I agree with the importance of the role of teachers in the process of acquiring beliefs about mathematics, the transmission model of teaching Kitcher seems to assume, (e.g. p119), is too crude, in my opinion, to explain an individual (or group) coming to know mathematics. Kitcher's discussions on warrants is worked in considerable linguistic abstraction and does not make the distinctions, which I think are crucial, between, for example, the parroting of a proof and an individual (or group) being convinced by a certain form of reasoning based on logic and structure. (The importance of this distinction was part of the content of chapter 5.) Thus it is not clear how the 'epistemological significance' of communities of practice is realised in a novice's knowledge formation. Obviously this is important for application to education, and as it stands his thesis is wanting. A further objection to Kitcher's theory, which I discuss below, is his explicit and thoroughly argued eschewal of the notion of mathematical object. Kitcher construes mathematical objects as part of Platonistic apriorism. It is the notion of a priori knowledge which he is most determined to defeat and if mathematical objects are only meaningful within a framework in which a priori knowledge is countenanced then 'mathematical objects' must go. As mathematical practice involves working with what seem to be mathematical objects (Kitcher acknowledges this) then his theory must (and does) offer a replacement. As I shall explain, I am not convinced that his theory of mathematical operations is ontologically different from a physicalistic theory of mathematical objects.

Kitcher tries to explain the relation between the perceptual and the mathematical at an ontological level, so his theory may be able to help explain the transition between pre-abstract manipulations and perceptions of 'infant mathematics' and the transitional abstractions of higher school mathematics. In other words, while the purpose of his thesis was to explain historical transition, an application in education may also be forthcoming. Indeed, a whole thesis could be written on how Kitcher's theory of 'rational transition' can or cannot be applied to people learning mathematics.

Chapter 6: Mathematical objects

To understand Kitcher's conception of mathematics, I think it is helpful to locate his work culturally and historically. His book was published first in the USA in 1983 and, as Kitcher acknowledges, it developed over some time, (1984, pvii). Kitcher was a graduate student in the history and philosophy of science at Princeton in the early 70s. So he was likely to be schooled in the formalisms developed in the first part of the century and still popular in the 60s, and it is also likely that he was part of the movement which critiqued these rigid conceptions of knowledge in the following decade. His 'Nature of Mathematical Knowledge' draws on the work he did in the 70s and I locate the work as a decade-long thought out reaction to the formalism of 60s mathematics. I would place Gila Hanna's 1983 (see 5.4.2) and John O'Neill's 1984 (see 7.2) theses in this same category. So, cultural forces acting on English language-medium philosophers included training in formalism and learning about its formal demise. This subsequently lead to a questioning of less precise forms of knowledge which were nevertheless constrained structurally, such as a priori knowledge. Kitcher's work is another attempt to crack the structure of a wider conception of knowledge which, like formalism, was seen to be rigid and failed to capture the nature of the knowledge as people experienced it.

6.6.1 *A priority and experience*

This critical analysis of formal, or abstract parts of knowledge may explain why the first part of Kitcher's 1984 book is taken up by refuting a priorism. If 'a priori' knowledge is independent of experience, then, Kitcher argues, it is explained in terms of a-psychologistic epistemology. But the growth of knowledge is a function of the living beings who discover or create this knowledge. Therefore epistemology - the theory of knowledge - to be properly explanatory, must be psychologistic. This is the sort of stance which Alvin Goldman was developing throughout the 70s and which cumulated in his 1986 book, which I discussed in chapter 5 and which Kitcher refers to as "the best available account of warrants" (p18). Kitcher associates a priorism with a-psychologistic epistemology in a strong sense: if any mathematical knowledge can be shown to be 'independent of experience' then, potentially, mathematical knowledge could be explained a-psychologically. This refutation of mathematical a priorism forces an

investigation into the nature of mathematical knowledge into accepting psychologistic epistemology. This means, crucially, human agency is entwined with mathematical knowledge.

We are able to imagine things that don't actually exist: unicorns and the Ether, for example. Yet there are domains in which these terms make sense and so are meaningful. As mythological creatures, unicorns are meaningful. The Ether is not meaningful as a contemporary scientific concept, but is meaningful as a concept in the history of science. In what sense is 'a priori' meaningful? Kitcher works very hard to convince his reader that mathematics is not a priori at all; he really wants to include the human, cognitive aspect as essential to his theory of mathematical knowledge and he feels compelled to rid the entire mathematical enterprise of the notion that there might be 'knowledge independent of all experience'. Kitcher does seem to set up a priori knowledge to be a metaphoric ideal rather than anything that actual humans can aspire to: "Rational uncertainty does not preclude knowledge, but it does rule out a priori knowledge." (ibid. p 43). In other words he refines the concept of a priori until it is like a winged horse - aesthetically beautiful and imaginable, but not obtainable.

This sort of interpretation of a priori knowledge makes it impossible to realise. The notion of 'knowledge independent of all experience' is interpreted to mean that there is no developmental, cognitive aspect to this knowledge type. For example, the idea that slave boys can have a priori knowledge of square roots and iterations without having had *any* experience of squares and their areas or of approximations; that all the knowledge is, chrysalis like, just waiting for the right conditions to hatch, is not compatible with scientific theories of cognition. This shows that the notion of a priori knowledge *can* be made unachievable by making the 'independent of experience' demands so strong. Indeed, Kitcher also concedes that "[e]xperience may be needed to acquire some concepts" (ibid. p 21). This observation surely assumes a that there is a cognitive pulse to the a priori which is consonant with the original conception of a priori. Kant describes how someone who could predict the consequences of undermining the foundations of his

Chapter 6: Mathematical objects

house a priori, “had first to learn through experience that bodies are heavy, and therefore fall when their supports are withdrawn.” (Kant, 1781, trans. Kemp-Smith, 1970).

In short, I understand why Kitcher is so keen to reject the a priori and I broadly accept his arguments against the a priori. Yet I feel that he restricts the concept of the a priori so much it becomes unrealisable and so loses the meaning of the term which Kant communicated.

Kitcher is driven to argue against a priori mathematical knowledge because of its a-cognitive feel. While I accept Kitcher’s rejection of the a priori knowledge in the forms he has presented it, I submit that there is something about the concept of a priority which does reflect a ‘mathematical feel’. Almost any basic mathematical fact would illustrate this point, with which I am sure Kitcher would concur. For example, the concept of circle that I have is ‘locus of points in a plane equidistant from a given point’⁵⁶. Given this conception of circles, I claim that my knowledge of elementary geometric facts like the properties of the angle in a semi-circle or the properties of tangents to the circle, are a priori. They are a priori in the sense that no experience I could have would change the propositions I assert as true while I held that a circle was a ‘locus of points in a plane equidistant from a given point’. This was of course recognised by Kant. He saw that ‘semi-circle and point-on-circumference’ do not analytically (by virtue of meaning) yield ‘right angle’. He tried to give voice to this recognition by coining the term synthetic-a priori for such propositions. I anticipate that Kitcher would counter that I am pointing to theoretical (ibid., p 55), rather than a priori, but I want to point out the phenomenological roots of the term a priori.

I have already used the term ‘a posteriori’ to denote the contrary to ‘a priori’. A very closely related, if not synonymous term is ‘empirical’. This standard philosophical term, which Kitcher uses freely, connotes knowledge based on perceptual experience. But the

⁵⁶ This has developed a long way from the circle concept I had as an infant, so experience has been required to develop the concept.

word 'experience' itself includes more than sense perception; experience of health, emotion and language, at least, also figure in our general experience of the world. If empirical knowledge is based on *sense experience*, is the larger class of knowledge based on *general experience* the referent for 'a posteriori'? I am not sure. As far as I can discern, the word 'experience' may be used by some writers to stand for perceptual experience - i.e. 'empirical' - or may stand for the wider conception⁵⁷, one cannot expect precise specification all the time. Indeed it can be argued that there is no distinction, that the empirical floods all our experience. In Donald Davidson's often quoted words "all we ever do is move our bodies" (e.g., Davidson, 1997H); our sensory and kinaesthetic experience is all we have.

Mathematics is traditionally seen to be the least empirical of all bodies of knowledge which abound in human societies. So, if Kitcher's thesis can be interpreted as constituting a case that even this body of knowledge to be empirical, then, in some sense all human knowledge is empirical. I think this reductionism to the empirical, given the two provisos that (a) mathematics is the 'least empirical' form of knowledge; and (b) Kitcher's thesis that mathematics is empirical, misses interesting questions about the nature of knowledge, even though I concur with the sentiment expressed by Davidson's aphorism. In

⁵⁷ For example, Kitcher seems to widen the notion of 'experience': "The appeal to linguistic understanding is not an a priori warrant, but, in the context of an experience which supports the propriety of the linguistic practice, it does provide knowledge." (pp94-5). Kitcher's careful specification that the linguistic understanding does provide knowledge within an appropriate context, shows that he is widening the notion of experience beyond the sensory (for a context which supports a linguistic practice must surely be at least partially social). But exactly to what is not clear. I am not sure how Kitcher thinks the knowledge claim is justified, so I shall exemplify his assertion, and leave the interpretation suggested by the example open for discussion. Consider rational and irrational numbers: To say I have a 'linguistic understanding' of these categories of numbers would usually mean that I can distinguish, define, operate with and represent these sorts of numbers. I do not think that I should be expected to prove any proposition which was presented to me. Nor does it mean that any statement I make about them - 'there are more irrational numbers than rational ones' say - is known a priori because I can appeal to my fluent linguistic capabilities. So I agree that appeal to linguistic understanding does not generally warrant belief. But, as this example illustrates, linguistic understanding *per se* is not enough for knowledge of rational and irrational numbers: in mathematics it is not sufficient to talk about concepts, action is needed too. But Kitcher says that in a suitable 'context of experience' my proposition 'there are more irrational numbers than rational ones' is knowledge because of the way that context supports the language. Grasping at straws, I can only suggest that such a context is something like a maths exam! And in this context a proof is the appropriate

mathematics, the question of 'abstraction' still persists even if all knowledge is decreed empirical; are abstractions linguistic concepts? functions of interaction between nature and thinker? or things-in-themselves? To say that language is 'just' neuronal firing and vibration, or that I exist within a material world, or that there are patterns in nature, is not very helpful in explaining the detail of mathematical knowledge.

6.6.2 *Mathematical objects*

While a question to debate is whether Kitcher's mathematical ontology is realist or not, Kitcher himself refuses to join either side: "The slogan that arithmetic is true in virtue of human operations should not be treated as an account to rival the thesis that arithmetic is true in virtue of the structural features of reality" (p 109). Indeed, he describes his position as "a peculiar form of constructivism" (in a Kantian sense) as well as having a "realist character" (p108); it may "be viewed as a type of realism" (p58). Nevertheless, the question of the ontological status of the stuff of mathematics remains. Kitcher manoeuvres between a Kuhnian interpretation of truth of mathematical statements varying with transitions in mathematical communities and a detailed analysis of the correctness, or otherwise, of parts of mathematics itself.

His theory develops the notion of mathematical ("idealised") action to replace those unobtainable 'mathematical objects'. We are to "switch from thinking of mathematics as descriptive of a realm of abstract objects to construing it as an idealised science of operations...[in particular] collecting and ordering" (ibid. p 138). But is this a mere semantical switch? I shall try to make the case that Kitcher does essentially assume the notion of mathematical object, particularly when discussing mathematics, even if he defines it away when presenting a philosophical theory. If this case can be made, then, I believe it follows that for any species of realist - and Kitcher is as 'social-constructivist'⁵⁸

contextualised linguistic response which 'does provide knowledge'. But this is just the glib received version of mathematical knowledge and adds nothing new.

⁵⁸ Kitcher's book was first published in 1983, some years before the term 'social-constructivist' had been coined. Indeed, in the recent (1995) 'Handbook of Epistemology', this term is absent.

as a realist can get - commitment to mathematical objects is unavoidable when immersed in mathematics discussion. I submit that it is this commitment to mathematical objects per se that is integral to learning mathematics, which I develop in 6.9. This is not required for an anti-realist: Dummett would take a semantic view of reference: language itself is the arbiter of truth.

I think that Kitcher is, despite his protests to the contrary, committed to mathematical objects for three reasons:

- a) Kitcher's ontology of operations is isomorphic to a theory of mathematical objects.
- b) The ideal agent construct is as abstract as any a priori knowledge.
- c) His detailed historical account of mathematical concepts - like 'limit' - which are intrinsic to the differential calculus, *is* an analysis of mathematical objects⁵⁹.

On a) Kitcher is trying to explain away the phenomenon of abstract entity - like many concepts in mathematics - by ontologising action. In mathematics, objects and actions are not ontologically distinct - 'the derivative of a function' involves calculation, for example. So, I think Kitcher still has mathematical objects, with a priori 'character' in his ontology, despite his claims to the contrary! Chapter 7 develops these ideas further.

On b) Platonism is problematic because, in that theory, mathematics requires ideal objects. Kitcherism seems analogously problematic because, in his theory, mathematics requires an ideal agent. Kitcher seems to want to endow mathematics with a 'larger than life' existence, despite his stated anti-a priori, or even anti-Platonic, stance. He tells us that it is because of our natural constraints "arithmetic owes its truth not to the actual operations of actual human beings but to the ideal operations performed by ideal agents." (ibid. p 109). What metaphysical streamlining is gained by denying 'mathematical objects'

⁵⁹ Of course it is always possible to deny the 'objecthood' of what one is discussing in language. The issue is of reference. When the idea of discovery or adaptation is incorporated, as in Kitcher's theory, it is not consistent to insist that mathematical objects are merely discourse-dependent entities.

Chapter 6: Mathematical objects

is lost by introducing ideal agents. Can his overall thesis of a "defensible empiricism" can be maintained without this key 'idealising' notion? My sense is that the notion of 'ideal' is not so very different from that of 'abstract' for his theory to stand distinct from a priorism.

Chihara has made a lengthy critique of Kitcher's theory (Chihara, 1990, pp 216 - 250). He makes the observation that accepting that school children know some mathematics does not imply that they have any conception of an ideal agent. Chihara claims that "According to Kitcher's analysis" the child who works out 12 times 7 by adding 70 to 14 "knows some very complex fact about the operations performed by some ideal agent" (Chihara, 1990, p 236). In Kitcher's defence, I would argue that Chihara is confusing domains of meaningfulness. It does not follow that young children who are at home with 'small' integer calculations should be able to give a philosophical account of those operations' validity. Nevertheless I concur with Chihara that the 'ideal agent' concept is really not comprehensible, for S/He must perform "all possible collectings of the members of [an] uncountable totality" (Chihara, p243) in "a medium analogous to time but far richer than time" (Kitcher, p 146). I do not think that Kitcher has managed to avoid the epistemological difficulties inherent in the notion of a priori by the 'ideal agent' construct.

On c) I shall now turn to a specific mathematical example and indicate how Kitcher's discussion indicates an ontological commitment to this 'mathematical object' qua object.

Despite his rejection of the concept of 'mathematical object', detailed analysis of an aspect of mathematical practice in Chapter 10, gives some insights into two specific mathematical concepts that could potentially be considered 'mathematical objects' - that of *limit* and *real number*. This last chapter of the 1984 book is concerned with an historical development of analysis from about 1650 to the end of the 19th century. Kitcher presents a 'rational reconstruction' of the development of the concept of limit which I review in order to assess the ontology his assessment implies.

6.6.2.1 *Kitcher on Newton and Leibniz's 'perceptual beginnings'*

Newton's concepts of fluxions and fluents formalised kinematic experience⁶⁰. Rates of change of motion can be perceived. Newton's insight was to distinguish rate of change from motion itself and to devise a method of working with these concepts that yielded perceptibly consistent results. My reading of Kitcher is that he would accrue 'reality' to these basic concepts of Newton. Mathematics has developed from this stage by (rationally) working on questions like: "Why are we entitled to make the assumption that the fluxions remain constant through small intervals of time? Why are we allowed to neglect some terms?" (p 233). For example, a perplexing property concerning the second question, is as follows: when a fluxion is multiplied by another variable, t , that is supposed to be "infinitely little" (after Newton, Kitcher p 233), the combined quantity $\dot{x}t$, vanishes! (For more details of Newton's method, see Fauvel and Gray, 1987, pp385-6.) The 'objects' fluxion and fluent are real, but their properties were not automatically understood.

Leibniz's approach was based on the concept of difference, rather than kinematics and originated, according to Fauvel and Gray (1987, p424) on his interest in logic and language. Specifically, it was Leibniz's notation that was his contribution as it "captur[ed] an underlying unity [and] made his discoveries easy to use" (ibid.). Perceptual reality was 'further away' from Leibniz's mathematical theory than from Newton's. My reading of Kitcher's presentation is that he construes Leibniz's calculus as algebraic, based on relationships, rather than perceptual-geometric, "[Leibniz] seems to shun the idea of endorsing an interpretation of [his calculus]" (Kitcher, op. cit., p 235). And he quotes Leibniz's adaptation of Berkeley's notion that the infinitessimals are 'useful fictions' (ibid., p237). The acceptance of Leibniz's method was in its generality: using his techniques a wide range of problems could be tackled. This is Kitcher's argument for the rationality of the acceptance of his method.

⁶⁰ Baron and Bos (1979) interpret Newton's fluxion and fluent concepts as scientific motion-concepts, following Galileo, Torricelli and Barrow.

Both semantic and perceptually based theories have to deal with the central concept of *derivative*, which can be referred to by other names and represented by several notations. The method of computing the derivative - for Leibniz and Newton did make the same calculation (using Kitcher's presentation) - involves some process notationally similar to dividing by zero. Dividing by zero provides no real numeric answer; it is a false method. While a false premiss implies any conclusion, surely the wealth of scientifically validated consequences is not all suspect? What is it that makes the initially dodgy-looking procedure valid?

The realist answer is that there was something, yet to be discovered, about the procedure. In this case it was the concepts of slope and of limit (à la Cauchy). The formal validation of the mathematical concept of limit had to be invented, to be sure. However, it formalises the troublesome 'littleness', essential in working mathematically with the problems of Newton-Leibniz calculus. The concept of limit provides ways to establish links between the empirical, (like tangents and ratios of differences approaching a fixed number), and more rigorous mathematical argument.

Clearly the community of those interested in the concept have to be convinced that it is a validating one, Kitcher's thesis is to identify rational ways that a community might make the transition to the acceptance of a new concept. Scientific efficacy is one such rational transition. In such a way, Kitcher's realism is akin to a dialectic of 'naturalised epistemology'.

6.6.2.2 *Kitcher's implicit mathematical objects*

Kitcher's use of 'geometrical' is interesting. The way he uses the term suggests a realism about the particular geometric entity under consideration; thus a realism about (some) mathematical objects. For example, he says of Newton that he uses a "geometrical conception of the limit" (p 237) which "is close to ... the modern definition of a limit" (p 238). Indeed, an anti-realist may use the terminology of mathematics as a coherent set of metaphors which might also include the 'meta-metaphor' 'mathematical object'. Talk about an object does not imply its existence. Kitcher's epistemology accepts a common-

sense realism about objects of perception, and asserts that mathematical notions have their genesis in operations on these. My suggestion is that limit, (as Newton reportedly construed), is a common-sense geometric object. It is in operations with this in the guise of "infinitely small parts of time" that Newton comes to his "ultimate ratio" (p 238). The geometric (in Kitcher's sense) limit exists in the perceptible world and is the object that imbues the symbolic mathematical notion with 'reality' (in a realist sense). To give another example: Kitcher observes that Cauchy's intuitive sense of continuity was not the same as the sense of his analytic definition of continuity. For the definition allowed Brownian motion type functions, continuous, but nowhere differentiable. Continuity, then exists 'in reality' and is connected to the idealised, mathematical notion.

The 'objecthood' of some mathematical entities Kitcher discusses clearly involves actions, as the two examples above illustrate. Actions mediate between perceptual, material objects and abstract mathematical ones. The actions, or operations, imbue the reality and the abstraction. As Kitcher says of his conception: "my picture of mathematical reality [is] constituted by the operations of an ideal subject" (p177). The view I shall elaborate further in chapter 7 is that, in some sense, some of these 'actions' can be identified with mathematical objects. In short, Kitcher's realism about objects and his action-theory which connects these experienced objects with mathematics, implies a realism about (some) mathematical objects (possibly construed as actions).

6.6.3 Platonist and non-platonist mathematical realism

The characters of the philosophies described by Maddy and Kitcher could fairly be captured as 'platonistic' and 'non-platonistic' realism respectively. This characterisation just gives a sketch because the term 'platonistic' is not really well defined and both Maddy and Kitcher rely on a Quinean 'naturalised epistemology' which privileges perceptual experience over conceptual coherence or mental intuition. Why I suggest that Maddy is a 'platonistic realist' is that she gives a unambiguous description of the nature of objects of mathematics. These ideal entities of her theory have something of the crystalline feel of Plato's Forms because of their set theoretic essence. On the other hand, the structure of Kitcher's idealised operations is less fundamental and so seems to be less

Chapter 6: Mathematical objects

rigid, less 'Form-like', for he bases his ontology on the epistemologically accessible manipulations and perceptions of ordinary objects.

6.7 An historical perspective on mathematical ontology

While Kitcher, a philosopher with interests in the history of mathematics and science, uses historical evidence to support his philosophical theory, Jeremy Gray, an historian of mathematics with philosophical interests, relies on the philosophical concept of mathematical ontology to express his thesis concerning the historical evolution of mathematical objects (Gray, 1992). Specifically, Gray's historical analysis can be interpreted as giving weight to the reality of mathematical entities. As I have discussed above, I think that Kitcher's historical analysis of mathematics cannot be distinguished from one which works within a frame which allows mathematical objects, despite his prior theorising against them. The vexed question of 'grasping an abstraction', which is part of the growth of mathematical knowledge, is dealt with in a different way by Gray in his essay "The nineteenth century revolution in mathematical ontology"⁶¹ (ibid.). The key point of Gray's essay is that the objects of mathematics, hitherto taken to be either intuitively known, or idealisations of physical phenomena, were fundamentally re-conceptualised as set theoretic or axiom-consequential entities. This constitutes an historical argument for the existence of (abstract) 'mathematical objects'.

"the new philosophy that underpinned these transformations [in ontology during the 19th century] ..was naïve set theory. It drove out naïve abstractionism and traditional Kantianism, and paved the way for its successors, the Formalist positions based on either logic or abstract axiomatics....The new ontology brought with it a new epistemology. The introduction of rigour in analysis is well known; I have attempted to show that the appropriateness of a proof received much more attention. This aesthetic awareness made sense at a time when the very objects of mathematics were themselves becoming more abstract." (ibid. p245).

Thus the reconceptualisation of mathematical entities produced an even stronger sense of mathematical object within mathematical practice. Again, the scientific efficacy of these notions supports their scientific reality against just being formal nominalistic or coherent linguistic items⁶².

6.7.1 *'Intuitionism' and 'naïve abstractionism'*

Gray argues that up to the 19th century, the reputed routes to mathematical knowledge were either Kantian-constructivist or 'naïve-abstractionist' and, so the entities of mathematics were constructions (also known as 'intuitions') or "idealisations of familiar objects" (p228).

Gray reminds his reader that 'intuition' has "several meanings": the Kantian conception of 'direct acquaintance', the result of familiarisation of the naïve abstractionist and the personal, psychological sense of "hunch" (p239). So, respectively, an intuition about, say, geodesics, could be (a) a direct acquaintance with energy minimising paths; (b) an abstraction from the perception of shortest distance (including shortest distances on curved surfaces like hills); or (c) the action-knowledge that to keep to a geodesic keep to the path whose curvature is the same as the surface you are on. The term 'naïve abstraction' is Gray's term to describe pre-twentieth century idealisations: "experience presents many objects that are nearly circular, and from them one abstracts the mathematical concept of a circle" (p228).

6.7.1.1 *Gray's ontological distinction*

I shall try to explain this through the geometrical episodes related by Gray, as geometry incorporates features of mathematics and mathematical thinking which are central to

⁶¹ I would like to thank June Barrow-Green for bringing this paper to my attention.

⁶² This is Azzouni's point about the importance of the efficacy of practice again (Azzouni, 1994 p 138).

Chapter 6: Mathematical objects

making a link between perceived regularities in the objects of our experience and those abstract objects of modern mathematical practice.

Gray argues that there was a profound re-conceptualisation of the contents of geometry in the nineteenth century, for example, what *straight line* referred to, changed. Naïvely it was the shortest path between two points in space. After the 'revolution', *straight line* meant geodesic (locus of points determined by specific differential equations) in the special space of constant zero curvature known, henceforth, as Euclidean space. The latter was axiomatically consequential rather than intuited or abstracted. This distinction serves as a good basis for understanding the notion of an abstract mathematical object for, with this understanding, the hitherto primitive term 'line' could not be simply abstracted from ordinary sense experience. Gray also quotes Nagel's (1939) analysis of projective geometry that illustrates an ontological distinction between naïve abstractions and axiomatic entities through the concept of projective duality. This duality, a consequence of the axiomatic system, shows that the projective geometrical terms 'line' and 'point' can be swapped (in tandem with 'concurrent' and 'colinear'). But the naively abstracted terms 'line' and 'point' are fixed in physical reality!

Gray explains the two senses of the collapse of the Euclidean conception of space. The first, due to Beltrami, showed that physical space could be described in a non-Euclidean way: "there is no unique mathematical abstraction" (p 234) of our familiar spatial objects. It no longer made sense to say 'line' was a primitive term (unless, like Hilbert, you realised you could just as well have called it 'chair'). There was no spatially given, axiomatic primitive, 'line'. Suppose the entire human population died out, bar Amanda and Evan. These two went forth and multiplied in the otherwise empty but still gridded city of New York, USA. Then it would be quite likely (!) that their descendants would have a 'taxi-cab' geometry as their first systemisation of space. The second collapse of the Euclidean conception of space, due to Riemann, was the liberating of geometry from physical space. This prompted an awareness that a reasonable way to understand the geometric nature of physical space was to empirically investigate which Riemannian manifold would be the most appropriate mathematical model. For domestic carpentry,

Euclid still serves well and is theoretically predicted to do so. In relativity studies, metrics for the models of four dimensional space are constructed on the basis of astronomical data. In either case, "no longer could it be argued that the term 'line' in geometry was a mere abstraction from a physical object" (p 235) or known intuitively. "There is no unique mathematical abstraction of familiar spatial objects available to our intuition" (p 234)

6.7.1.2 Abstract objects and 'levels' in mathematical ontology

The historical distinction argued for as an historical thesis by Gray is mirrored by the two-tiered ontology Maddy discusses. "Naïve abstraction" corresponds to the lower tier in Quine/Putnamism and 'intuition' corresponds to the lower tier in Gödel's scheme. Maddy's theoretical link between these two levels is the set theory of modern mathematics. Her argument is that some objects of set theory are within the physicalistically experienced world.

While Maddy's theory provides an ontology, it fails to provide a comprehensive epistemological explanation. Perhaps Maddy can convince her reader that numbers are properties of perceptible sets, thus offering a theory about how these entities are known. However, the objects of geometry, like geodesics, are different in kind: the concept of geodesic is not equivalent to its set theoretic equivalent; theorems about them do not usually involve set theoretic proofs. The increasing requirement for conceptual proofs, Gray argues, is one of the consequences of ontological discontinuity. In other words, in mathematical practice it became more important to have a knowledge of these abstract objects which was not 'just' through set theory. The set theoretic connection may guarantee the reality of their ontology, but another explanation is required for the guarantee that these abstract geometrical objects are knowable (in the sense of what constitutes a proof about them).

6.7.2 *Gray on mathematicians' epistemological evolution*

Gray suggests that there was a change in epistemology, consonant with the 'revolution' in ontology: "the new conceptual and aesthetic criteria have often achieved paramount position at the level of explanation overthrowing mere calculation as the best criteria for truth." (p 239) Perhaps this sort of warrant based on the properties of abstract entities was only employed by a few, the great, mathematicians of the time of the 'revolution'. In the twentieth-century, much of mathematics requires this concept-object type of reasoning. (An exception, perhaps, is mathematics for computing in which algorithms are the ontological and epistemological foundation). And without an ontological commitment to these object, the notion of a conceptual proof justifying a property seems inconsistent.

The great mathematicians of the nineteenth century changed their ontological foundation from 'idealisations' to 'abstractions' and this change was epitomised in their work. In terms of epistemology, let me suggest the following, which may have some educational analogy: their orientation to proofs based on the conceptual, set theoretic and axiomatical mathematics (rather than computation or manipulation) of their 'revolutionary' ontology was not achieved in a discrete a way as the notion of ontological discontinuity suggests. Rather their epistemological change evolved and continued to retain warrants of all types - this is Zheng's recognition that there is realistic as well as formal mathematical truth (see 6.7.4 below). These are difficult to prise apart when involved in the practice of mathematics.

Mathematics in the twentieth-century is not often just concerned with idealisations (perfect circles and so on) of easily perceptible material objects. Pure mathematics and applied mathematics alike deal with 'abstract' concepts which have been termed 'mathematical objects'. In mathematics in education, idealisations of easily perceptible objects is an important concern. For, like Kant's man who could predict the consequences of undermining the foundations of his house a priori, children develop a significant part of their knowledge, even of 'a priori-feeling' mathematics, from their perceptual experience. This 'naïve abstractionism' is alive and well in most school classrooms. Do students of mathematics have to go through a process analogous to the historical

development before they are conceptualising 'abstract' objects? My belief is that being clearer about ontological distinctions and the epistemological changes that they require, should put us teachers in a better position to help students make a more effective transition.

6.7.3 Other historians' of mathematics dichotomies

Caroline Dunmore's thesis (Dunmore, 1992) is that the objects of mathematics continue in time but there exist meta-level revolutions in mathematics. This is another two-tiered theory: the base-level are the objects of mathematics and the meta-level is the conceptions about the nature of these objects. Dunmore's analysis lends weight to the idea that there can be historical continuity construed between the objects of perception which have been 'naïvely abstracted', and the objects of current mathematics. This allows a transition from a physical conception of putative mathematical objects to a conception of mathematical objects which - in Plato's translator's words - are not to be 'confined to visible or tangible objects'⁶³. The limits, lines and series which were 'naïvely abstracted' mathematical objects, remain as mathematical objects after the 'revolution'. But they have changed their character. They have now been reconceptualised through various mathematical strategies. For example, axiomatisation changes the nature of the term 'line' and set-theoretic underpinning of number theory changes the character of what numbers are. Incorporating Dunmore's ideas with Gray's allows a continuity of meaning of some linguistic labels, like 'line' and 'circle', for example, but a shift in their reference and in the warranting of their properties (as described in chapter 5).

Zheng Yuxin's (1992) distinction between types of mathematical truth may also be helpful. He calls them "realistic truth [which] means conformity or agreement with reality [and] formal truth ... [in which] mathematical statements are truth about the corresponding structure" (Zheng, op. cit. p179). Neither conception of truth is to

dominate, but rather “a synthetic view should be adopted” (p180). The idea that these types of mathematical truth are 'fractally intertwined' suggests that, given the distinction Zheng offers, when discussing the truth value of some mathematical proposition there will be formal and realistic notions of truth within any constituent part of the proposition (proportions would vary given the propositions, of course). I think that this is likely even for propositions in elementary mathematics, e.g. the angle in a semi-circle is a quarter turn: this truth has both realistic and formal components. Within Zheng’s conception of realistic and formal truth, the proportion of ‘realistic truth’ decreases as mathematical propositions become more abstract. While this seems reasonable enough, Zheng does not offer a way of apportioning measures of truth-type to any given proposition.

As the shift towards Zheng's formal truth increases with the more abstractly formulation of mathematical entities, in the limit, it could be argued, there is not a trace of the realistic remaining. This stage is for Kitcheresque ‘ideal agents’ only! Even highly mathematical human subjects retain realistic links, perhaps as psychological, rather than epistemological sense: their concept image has a perceptual dimension but they are still able to present to themselves and their wider community a proof which does not depend on the perceptual. These ideas of Zheng’s on truth in mathematics reinforce some of the ideas I expressed in chapter 4.

6.7.4 *Historical evidence for mathematical objects*

Gray's thesis, together with those of Dunmore and Zheng, lends more weight to realism in mathematics: because their historical analyses of the revolution in mathematical ontology give credence to the idea that there are mathematical objects which are both applicable in the physical world yet are not constrained by it. Their abstract features are not like that of approximation (as being close to a circle) but due to properties of relations. These properties can sometimes be detected in the physical world and they can also be

⁶³ [The study of arithmetic] "draws the mind upwards and forces it to argue about pure numbers, and will not be put off by attempts to confine the argument to collections of visible or tangible objects" (Plato, 'The

formalised. Consequences of these formalizations constitute the results of modern pure mathematics and consequences of investigations of the properties constitute the results of modern applied mathematics.

Mathematical ontology's fundamental change required increasing rigour in justifying theorems, but this did not break with prior conceptions of mathematical objects⁶⁴. However, this tendency to rigour was not as discrete as the ontological changes. In other words, epistemologically there is a continuous thread. This continuity enables the possibility for teaching and learning modern abstract concepts starting from a physical, experiential base rather than a purely linguistic, social one.

6.8 Mathematical realism in the classroom?

In sections 6.4 to 6.6 I have presented conceptions of mathematics which the authors consider, in some sense, realist. The acid test for the realism, or anti-realism, of a theory of the nature of mathematics is the conception of truth it entails, (see chapter 4). On one side is the anti-realist conception of truth: propositions are true by convention or because of linguistic coherence. Verification is a question of fit, of meanings corresponding to the agreed group perceptions. This is an epistemic conception of truth. On the other side, realist conceptions of truth grandly claim 'a proposition is true because of the way the world is' independent of anyone's knowledge of that truth. These are nonepistemic conceptions of truth.

The difficulty with the anti-realist conception is ontology and the difficulty with the realist conception is epistemology. This is a vast simplification, but what I want to indicate is the resistance of anti-realists to assert the existence of anything outside their group (the 'objectivity is social' view) and the problem realists have to answer when

Republic' trans. Lee, 1955, page 293).

⁶⁴ A way which may be successful in forcing students to break with their naïve abstractionist views of number, line etc. is with an increased emphasis on axiomatics. Balacheff in France and Marriotti et al. in Italy have recently said that they are working on pedagogical representations of these ideas (Geometry working group discussion PME 1997).

confronted with the difficulty of accessing - i.e. having reliable knowledge of - the world, the state of which determines the truth value of a given proposition.

Mathematics education is naturally more concerned with epistemology - the nature of knowledge and how knowledge develops - than ontology - the nature of what it is students are to learn about. This is because of the dominant classroom-based practical side of the discipline. It is not surprising then that, as the emphasis on knowledge development is prioritised, much contemporary theory is based on an anti-realist notion of truth. Indeed, the ostensibly ontological question 'the students are to learn what knowledge?' can be answered by 'knowledge of what we (in the students' community) think fit to teach them'. Curriculum designers, benign or otherwise, decide what students are supposed to know, and if the students do not give 'recognised' responses, their work is wanting⁶⁵. I do not find this satisfactory as a theoretical justification of entitlement to knowledge, although pragmatically it is likely to happen. This is one reason why I have put the case that attention should be paid to ontology, as well as to epistemology.

The other part of my argument that education should be concerned with ontology as well as epistemology comes from thinking about mathematics specifically. Just as education must be concerned with epistemology, mathematics must be concerned with ontology. As I have just related, this has been argued by Gray (1992, and also 1997) who examines the importance of mathematical ontology to mathematicians at different historical times, (6.7). People struggling to make sense of a-sensory mathematics have work to do to find what is real and what is a chimera. Learners, like fully fledged mathematicians, need to know what is and what is not in mathematics, which infinite series are functions and which do not converge, for example. But unlike the mathematicians who 'put these concepts on the mathematical map', students *can* adopt conventions and routines without the ontic struggle the initiating mathematicians historically experienced. This

⁶⁵ A non mathematical example of this occurred in 1980 when a group of feminist Oxford undergraduates answered questions on their English finals using feminist analysis. Their essays, considered otherwise first class, were all awarded seconds as they had used the 'wrong theory'; the dons' 'régime of truth' did not extend to gender oppression.

phenomenon, of taking a concept on trust, working ‘as-if’ it existed or made sense, is probably an inextricable part of regular school, or other, learning. But this acceptance of an authority’s decree about ‘what is’, or what to do with ‘what is’, is conceptually different from a personal ontological commitment. In learning about negative numbers, for example, a pupil might accept that ‘a minus multiplied by a minus is a plus’ for ‘reasons we need not discuss’. For knowledge of negative numbers, she requires a sense of the reality of these numbers and a rationale for their properties based on ‘what they are’. This is not easy to achieve, as may be judged by examining the pupils’ work discussed further in the next section, 6.9.

I hope to have explained, here, why a philosophy of mathematics suitable for learning mathematics is worth predicating on realism in mathematics. Before dealing explicitly with ontological commitment, I need to clarify mathematical ontology for mathematics learners, based on the realist theories I have presented. After that, with a theory of ‘what is’ in higher school mathematics, I shall explain the notion of ‘ontological commitment’ and discuss its importance in learning mathematics.

6.8.1 Mathematical ontology for higher school mathematics

What ontology, what theory of what exists, can I now specify for higher school mathematics? I cannot specify what does exist - that is the mathematical enterprise - an ontology is a theory, or a way of conceptualising, what does exist. At the school level, a theory of what exists in mathematics must, pragmatically, be interwoven with a theory of what can be known, otherwise the educational thread is missing. So, from my ‘experiential stance’, I want to base the higher school mathematics ontology on ‘experience’. But ‘experience’ per se is far too wide a concept. It can mean experience of anything from mystical insight to that of information from the senses. This is why I have looked to a philosophical tradition which delineates a notion of experience which is fruitful and specific: that of contemporary realists like Maddy, Kitcher, Resnik and Bigelow. These philosophers all subject their ontologies to broadly the same ‘naturalistic’ epistemology - what we know is within science and in science the ‘buck stops’ with sense

Chapter 6: Mathematical objects

perceptions. This Quinean view is in the tradition which started with the British empiricists in the 17th and 18th century.

The challenge for a philosopher of mathematics is to explain 'mathematical abstraction' from this epistemological constraint. For higher school mathematics, a theory of what exists mathematically, must have this sort of experiential dimension within its epistemological counterpart. The amalgam realist theory which I suggest starts with a non-metaphorical sense of 'mathematical objects'. That is, the theory of 'what exists' states that there are some mathematical things. These things are categorised as 'objects' because they share with material objects the feature that they are 'objective' in the ordinary language sense of the word. These mathematical entities are a function of the interaction of cognizing beings, which perceive in specific ways, and the physical environment. Uniqueness is not claimed but objectivity is. The type of perceptions the cognizing beings have will limit the patterns perceived from the environment. For example, human beings can perceive discreteness and continuousness and distinguish these. Thus, from this attribute the mathematics of number and measure can evolve. Clearly, any such 'evolution' must be supported in language and culture for its survival.

The next question is what is the nature of the abstractions which are the mathematical objects human beings can know about? Kitcher's 'operations' and Maddy's 'set-theoretic perceptions' are examples of such. In each case they are epistemologically transparent mathematical objects⁶⁶. What then do I claim exists in (higher school) mathematics, which is consonant with my practice as a teacher and a 'this-level' mathematician? Because I want as much choice and potential as possible in working with students, I want to take the union of the classes of abstractions described by the professional philosophers as the contents of the ontology. By basing an ontology on prior theories with well established naturalistic epistemology, I should ensure that the entities thus specified are knowable, in the sense discussed in chapter 5. There are two main objections to this

⁶⁶ Even though Kitcher does not like the term 'mathematical object' I have argued that his physically based realism does not actually avoid his mathematical entities 'object-ness' - i.e. 'objectivity'.

approach: (i) ontological redundancy; (ii) potential ontological incompatibility. For (i), I am not worried by an over abundant ontology, even though ‘ontological economy’ is considered a virtue by philosophers (see, Alston, 1958/64). As a teacher I like the idea of plenty of starting points! But are they, (ii), compatible? Yes, because they all have to have a human-perceptible environmental basis. There is more to discuss here, but the detail of this claimed compatibility would take me beyond the scope of this work.

So, more specifically, what can be made of the philosophers’ theories with respect to school mathematics?

- 1) Firstly, that mathematics is neither ‘pi in the sky’, nor mere ‘formal fictions’, nor just a ‘cultural artefact’. The physicalist premiss is that there are patterns inherent in the physical world. And this sense of realism allows some ability to act on, or otherwise perceive, these patterns, properties of these patterns, or relationships between these patterns. Mathematical knowledge results (or can result) from these actions and perceptions.
- 2) Secondly, some people notice these patterns, encode them linguistically and symbolically, and some also point them out to others. For example, consider this activity for infants: they play with three teddies, each of whom has a hat, a chair and other possessions. Whatever the infants might make of it, this activity is a pedagogical representation, (a ‘pointing out’) of the physical pattern called ‘three’. At the higher school level, the pattern of (countable) infinity is often problematic, for, unlike triples, these infinities are not directly perceptible. However, a recurring action *is* perceptible. For example, I cut a square cake into two rectangular halves, I cut one of these halves into two quarter-sized squares, I cut one of these squares into eighth-sized rectangles, and so on. To be sure, the idea of cutting forever could seem ridiculous to some: ‘I’d die before I’d finish!’, ‘The cake would all be crummy’, ‘wouldn’t you get down to an atom eventually?’ Mathematics is an imaginative and playful venture (e.g. 6.5.1.1) as well as a practical one.

Chapter 6: Mathematical objects

- 3) Thirdly, teachers can only 'point out', not force another's imagination. So there are no guarantees that a student's cognitive awareness of the fact that $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1$ will have been raised by the activity of contemplating the pattern of the action of repeated halving. The point is that 'halving repeatedly' is a physical pattern.

Kitcher and Resnik both recognise the significance of action in developing mathematical knowledge. I shall take this idea further in Chapter 7, where I want to investigate the idea that certain actions are, themselves, mathematical knowledge; I shall put the case that not only does 'action enable', but also 'action is'.

In order to talk about mathematics properly, some specification of how abstraction is achieved is due. Most of such a discussion would fall into the psychological domain, which is outside my brief here. However, as I am trying to work within a naturalistic, and hence psychologicistic, epistemology, it is appropriate to point to some psychological theory to illustrate that the transitions to abstraction are possible. Again this task could be researched in depth, but here I do little more than quote two theoretical psychological approaches. The first is from Zoltan Dienes (1970) who presents an outline for constructing teaching plans in terms of stages. These stages are designed to promote abstracting in the pupils who experience this teaching. The stages are:

FIRST STAGE - free play within the environment; **SECOND STAGE** - realisation of constraints in the situation; **THIRD STAGE** - recognition of structure; **FOURTH STAGE** - representation of structure; **FIFTH STAGE** - linguistic encoding of the representation of structure; **SIXTH STAGE** - axiomatization, rule specifying and deduction. (Adapted from *ibid.* pp6-9 and pp53-4).

The other is due to P. van Hiele and D. van Hiele-Geldof, (1958), which is well known for the stages of geometric abstraction⁶⁷. This theoretical psychological developmental picture was originally conceived in five-stages which are:

FIRST LEVEL - recognition or visualisation of physical attributes perceived from the whole; SECOND LEVEL - analysis of properties of these 'wholes'; THIRD LEVEL - ordering, systematization of those properties; FOURTH LEVEL - receptivity to deduction given 'properties'; FIFTH LEVEL: production of rigorous deduction. (adapted from Hershkowitz, 1990, pp72-3).

Both of these stage models, conceived for different purposes, are scientific theories which contribute to the naturalised epistemologists' project of understanding what knowledge is through science. They are also similar in their accounts of how abstraction is achieved. As scientific theories, they are subject to empirical tests and criticisms. For example, Hershkowitz reports that the van Hiele levels "fit the described hierarchy, with a few exceptions" (p73). My purpose in quoting them is to mark that the philosophical framework which I have been developing does have a counterpart in scientific theory on which this philosophy theoretically relies.

6.9 Ontological commitment and learning mathematics.

In this section I shall discuss the notion of ontological commitment and whether, or to what extent, learning mathematics involves ontological commitment to mathematical objects.

This section, which is the main justification for this sub-thesis part (ii): 'individuals require ontological commitment to some mathematical objects for mathematical knowledge'. It is in three main parts: in 6.9.1, I present some philosophers conceptions of the notion of ontological commitment; then, in 6.9.2, I investigate the relationship

⁶⁷ Mason (1996) offers an interpretation of the 'van Hiele' levels which generalise their use from their original geometric context.

between ontology and ontological commitment; in 6.9.3 I return to some specific cases of realism and pedagogy which are intended to illustrate some of the more theoretical ideas discussed in 6.8.1 and 6.8.2.

6.9.1 *What is 'ontological commitment'?*

The term 'ontological commitment' is used in philosophical discourse to connote reliance on 'what there is'. Because there are different possible consistent and coherent answers to the question 'What is there?', alternate ontologies are conceivable. And 'ontological commitment' seems to assume an ontology. But in the process of learning (in particular, mathematics), this apparent tautology - that ontological commitment presupposes 'ontological items' - is not quite as straightforward as it seems. Before taking this point further, I want to recap on the ontologies of the three main philosophies of mathematics, realism, nominalism and conceptualism.

In the case of mathematical realists, locating their ontology is fairly straightforward. The ontology consists in objects of mathematics. These are the entities of the discourse of mathematics as it is practised. For physicalist mathematical realists, these entities have some kind of perceptual link⁶⁸ to the objects of either common-sense realism or basic science.

In the case of nominalists (like Field), their ontology consists in space-time particulars. Strictly mathematical entities do not exist, they are 'formal fictions'.

Turning now to conceptualists, the situation is more complicated. This is because the phrase 'ontological commitment' can be interpreted semantically or existentially (generally) and these two interpretations are close in a conceptualists' scheme. As well as that, there are different levels of interpretation in each case. Consider, firstly, an

⁶⁸ As described above, Maddy's link is through the perception of sets, Kitcher's link is through the manipulation of objects. etc.

existential interpretation of 'ontological commitment', there are two levels of interpretation:

either

0, 'mathematical objects exist independent of discourse'.

or

1, 'mathematical objects exist within discourse'

The first statement, indexed 0, is a realist's assertion⁶⁹, rejected by conceptualists. But, clearly, it would not be inconsistent for a conceptualist to claim ontological commitment to a metaphoric construct in the language game in which he wished to participate, i.e. to accept statement 1. Now, consider a semantic interpretation of 'ontological commitment'. In this interpretation we still have 1, 'mathematical objects exist within discourse'. The other level of interpretation is

2, 'the meaning of 'ontological commitment' (in mathematics) is that mathematical objects exist within discourse'.

Now 2 is applicable to realist as well as conceptualist discourse. This just says that realists cannot sensibly develop knowledge without a discourse, as if that was news! I think it is possible to put a wedge between realist and conceptualist by inserting the qualifier *only*:

2*, 'the meaning of 'ontological commitment' (in mathematics) is that mathematical objects *only* exist within discourse'.

If 1 and 2* are taken to characterise conceptualists' notion of ontological commitment, then such commitment can only be relative. In particular, the ontology of common-sense experience - Maddy's 'medium sized objects' - are not foundational to the concepts of the

Chapter 6: Mathematical objects

discourse, except as relative items themselves. So, in terms of philosophical (*not* psychological!) analysis, a teacher cannot communicate object hood about, say, triangular numbers, by getting the pupil to manipulate or perceive medium sized objects, instead she establishes linguistic conformity about triangular numbers on the basis of a ‘taken-as-shared’ discourse about counters, numerals or whatever.

What do philosophers have to say about ontological commitment? I briefly present some ideas from the literature:

6.9.1.1 Alston

Wm. Alston's 1958 essay, (Alston 1964), develops the idea that ontological commitment is ‘beneath’ language. Alston shows this by arguing that the translation of ordinary language statements, which naively suggest ontological commitment, into a philosophic register does not refute ontological commitment.

“in any ordinary sense of the terms, whether a man admits (asserts) the existence of possibilities depends on what statement he makes, not on what sentence he uses to make that statement.”

“This means that assertion of existence, commitment to existence, etc., does not consist in the inflexible preference for one verbal formulation over any other.” (ibid. p254)

The tone of his paper is that ontological commitment is to be avoided because of the difficult questions ‘abstract’ entities raise (ibid. p 256). The central thesis of the paper is that philosophers’ attempts to ‘translate’, i.e. paraphrase, sentences of the form ‘There are Ps’ to sentences of some other form, does not, as was desired, help avoid ontological commitments. These philosophers have conflated terminological problem with the existential problem.

⁶⁹ Strictly ‘the truth of mathematical propositions exists independent of discourse’

However, Alston concurs with the essence of Quine's definition of 'ontological commitment' (p 253), which in an ontological commitment rests ultimately on common-sense and science (see 6.6.1.1), to which I now turn:

6.9.1.2 *Quine*

Quine's well known essay, 'On what there is', (Quine 1953), puts forward two relevant important points: Firstly, that a word can have meaning without its referent existing (paraphrased from *ibid.* p189); secondly, that "Our acceptance of an ontology is, I think, similar in principle to our acceptance of a scientific theory ... we adopt ...the simplest conceptual scheme into which the disordered fragments of raw experience can be fitted and arranged," (*ibid.* p194).

Quine argues the first point using Russell's theory of 'singular types'. This theory explains the mechanism whereby names can be replaced by descriptions. This enables meaningful use of names without presuming commitment to the existence of that which was named. In this way I might be able to name some negative numbers and give some description of them without ontological commitment. Indeed, minimising ontological commitment seemed an important aim for these mid-century philosophers. However, I shall claim that some form of ontological commitment is intrinsic to the practice of mathematics in 6.9.2. To help justify this claim, I present samples of some students' work, in 6.9.3.2, which can be construed as illustrating 'naming and describing' without ontological commitment.

As I discussed in the section describing Maddy's set theoretic realism, Quine is a 'naturalised epistemologist' who regards science as 'real' but mathematics as purely semantic. He holds with a common-sense and scientific ontology (or ontologies?!), but does not (by 1969) include mathematics within these. Now in this 1953 paper he says:

"One's ontology is basic to the conceptual scheme by which he interprets all experiences, even the most commonplace ones. Judged within some conceptual

Chapter 6: Mathematical objects

scheme - and how else is judgement possible? - an ontological statement goes without saying, standing in need of no separate justification at all." (ibid. p189)

My assertion is that a mathematical ontology is intrinsic to the way I interpret experience. Discreteness, continuity, chance, etc. are part of my 'conceptual scheme'. However, I expect that Quine would counter that I was 'merely' familiar with mathematics-as-language. Ironically, Quine uses a mathematical example to illustrate how he sees the possibility of ontological commitment: "the *only* way we can involve ourselves in ontological commitments [is] by our use of bound variables" e.g. "*there is something which is a prime number larger than a million*", (his italics p191). Despite the mathematical content, I am with Alston in not being convinced that the form of words (or symbols) is the arbiter of ontological commitment.

6.9.1.3 Devitt

Michael Devitt is a contemporary realist philosopher whose sense of ontological commitment is different from Quine's and Alston's. To begin with, Devitt distinguishes ontological commitment from language facility, but notes that discussion about these positions is inevitably semantic:

"Merely stating, for example, 'Common-sense entities exist' does not ontologically commit you to those entities. Commitment depends on the truth conditions of the statements we accept; we are committed only to the entities which must exist for our statements to be true. The ontological question becomes clear only when we move into the metalanguage and consider this semantic question. The disagreement between the realist and anti-realist is not over statements like the one above but over how such statements should be understood. So the disagreement is a semantic one." (Devitt, 1984, p40)

This passage shows how Devitt's realist stance becomes entwined in his concept of ontological commitment. What we assert is true, not what we can speak about, is the essence of our sense of what there is. For "it is a truism that a theory must be presented to

us in language... But this ... supplies no reason for supposing that we must move to a semantic theory to determine the ontological commitment of our object theory.” (ibid. p42).

6.9.1.4 Harré

Devitt’s conception of ontological commitment is explicitly a realist one which relies on his correspondence-like theory of truth. A theory, which claims to be realist but avoids truth issues is given by Harré (1986, p74). In this conception ontological commitment to common-sense objects, (those in ‘Realm 1’), is assumed and imaginative possibilities about these objects is also countenanced. The problem here is the shift between the common-sense realism of Realm 1 and the imaginative projections (into Realms 2 and 3) from the common-sense realm. The types of ontological commitment seem different according to the realm: roughly speaking, in Realm 1 ontological commitment is ‘beneath language’, in Alston’s sense, but in Realms 2 and 3, ontological commitment is semantic. The realist perspective is, as far as I can currently tell, served best by a conception of ontological commitment along the lines of Devitt’s.

6.9.2 *Ontology and ontological commitment*

To begin with, consider the question: ‘Does doing mathematics require ontological commitment to mathematical entities of the mathematical activity?’ This statement could be read as tautological, but suppose it does have content. There are several ways in which it could be answered in the negative:

- ◇ at the moment of discovery there is no prior ontological commitment (although, doubtless, plenty of psychological commitment)
- ◇ doing mathematics does include formal manipulation (although fluency with the symbols is required)

Chapter 6: Mathematical objects

- ◇ learning mathematics involves doing mathematics and so ontological commitment is not a discrete occurrence (despite the psychological feeling of 'got-it-ness' that can occur at discrete times) but a dialectical process developing over many years.

It is possible to work on some mathematical activity or investigation without ontological commitment to *all* the entities involved. Some such entities may be in the process of being grasped and some may be used instrumentally. In this sense, developing ontological commitments can take a long time. Do the contents of these commitments just 'feel' as though they are real? This comes down to asking whether mathematical ontology is a question for psychology. I think that this is an interesting, but different, question from the one on which I am working. Infants learn about physical objects, and their properties, such as permanence in general and the unyieldability of the kitchen table in particular. How they do this is a psychological question. Mathematics students learn about mathematical objects, and their properties such as abstractness in general and the necessary right angled-ness of the angle in a semi-circle in particular. Again, how this is achieved is a psychological question. In both cases, there are philosophical questions about the existence and nature of these 'objects' and practical questions about how to help someone to understand their properties.

Having tried to distinguish between the psychological and the philosophical, I now return to the conceptual issues of

- ◇ the nature of ontological commitment, assuming a realist ontology, 6.9.2.1.
- ◇ the nature of a suitable ontology for mathematics learning, given the desirability of ontological commitment, 6.9.2.2.

6.9.2.1 *Consequences of adopting a realist ontology*

If we assume realism - of a physicalist variety, at any rate - then it follows that there is a (non-metaphorical) sense of existent object. For some of these objects, an individual can have an interaction with them independently of a discourse. A given individual will surely require a supportive community to confirm and publicly label these objects, but the

assumption is that these objects have independent existence, although discussion about them clearly cannot. The root of individuals' ontological commitment to mathematical objects is the perceptions, operations and predictions they can achieve with these existent objects. At first these objects are likely to be 'medium sized objects'. Subsequently, the 'objects' are the relations and patterns which the individual can perceive, operate on or predict something about from these primary givens, and so on. This conception is consonant with Hacking's, reported by Harré, who is also "in substantial agreement":

"Like [Robert] Boyle, Hacking thinks in terms of projects which involve *doing* something with clusters of entities, the effect of which is to 'interfere in other more hypothetical parts of nature' (Hacking, 1983, chapter 16). A hypothetical entity becomes real to us when we use it to investigate something else" (Harré, 1986, p51).

The individual's commitment to an ontology which includes 'hypothetical entities' is a function of actions and operations on ontologically prior 'clusters of entities'.

6.9.2.2 *Experiential learning of mathematics and ontology*

As I discussed in section 6.9.1, the notion of ontological commitment, in and of itself, does not imply a realist ontology. What I want to make a case for now is that, given certain further assumptions about the nature of learning, a realist ontology is an appropriate one for learning higher school mathematics. It is not possible to prove this assertion logically, all I can do is advocate it.

The premiss that I want to take is the empirical, psychological result that learning is a function of an individual, supported by a shared language and culture, not a function of knowledge being 'transmitted'. I do not think that this is a contentious assumption, but I note that it is psychological, not philosophical⁷⁰. We learn by experience. Teachers facilitate learning effectively by working with students' prior ontological commitments.

⁷⁰ The related philosophical issues include queries à la Hume about the 'external world' and in what sense socially constructed knowledge is 'objective'.

Chapter 6: Mathematical objects

This is a quite general statement and applies to discourse centred conceptualists as well as independently-existing-entity centred realists. The example of 'volume' can illustrate ontological requirements in either ontology: If 'volume' is to do with hearing something, it is not also related to $l \times w \times b$; the discourses are different. In the other case, suppose a student has not conceptualised the concept of 'amount of space' as different in 2 and 3 dimensions. Then the specific 3 dimensional concept, with the associated formula for boxes, $l \times w \times b$, is likely to be misunderstood.

Now experiential learning does include actual, physical interaction with material things in and of the world. Action can happen without language. And, from Alston's paper, ontological commitment is 'beneath language'. Because of this seemingly simple observation, I advocate physicalist realism as an appropriate ontology for learners of mathematics.

6.9.3 Back to pedagogy: negative numbers

This section links some of the theory with practical issues. Firstly, I look at one of the philosophical theories I have discussed and describe possible ways that this theory could underpin pedagogical approaches. I have chosen to use Maddy's theory for this exercise because of its more precise ontology. Secondly, I present and interpret some students' work with regard to the notion of ontological commitment.

6.9.3.1 Negative numbers and set-theoretic realism

Maddy's epistemological access to mathematics is based on direct perception and scientific experiment. How might her 'set theoretic realism' transfer to the mathematics classroom? There have been curriculum projects which could be interpreted as trying to do this. As I describe below, the School Mathematics Project did try to base a curriculum on set theory. Hanna's critique of the 'new math reforms' of the 1960's was discussed in 5.5.3. Hanna's major point is that formalist precision tends to mitigate against fluent understanding.

Can set theoretic realism answer students' questions like "What is 'minus one'?"? It can at a philosophical - or, more precisely, ontological/metaphysical - level. 'Minus one' is the additive inverse of the generator of the ring of integers. The generator is 1 is perceptible as the number property of the singleton set; the rest of the explanation rests on set theoretic construction of the ring of integers. Such an explanation might satisfy Maddy but is unlikely to be sufficiently related to the properties and function of -1 to satisfy students.

Of course, there is no necessary connection between a pedagogical representation of a mathematical concept and its strict set theoretic definition. A teacher who wants to explain the nature of mathematics in physicalist terms, might choose to explain the connection between mathematical abstraction with scientific reality based on set theoretic realism, but whatever their ontological position, their pedagogical representations are bound to include linguistic or contextual detail. Nevertheless, I want to consider two well known pedagogical representations of 'minus one' and discuss whether they are compatible with a set theoretic realist's position.

To start to teach the concept of negative numbers, we can ask questions like (a): 'how can 'negative one' be perceived?'; or (b): 'of what set is -1 , the number property?'

On (a) To perceive -1 , or as here -2 , is to be able to (non-metaphorically) perceive a lack. In the film 'Stand and Deliver', (Warner Brothers, 1988), the teacher introduces the notion of negative numbers through the concept-image of a hole: you dig 2 feet down, you fill 2 feet back up until you get to zero, the base level. This can serve as a perception of negative two. The 2 feet down-ness can be felt and seen; the lack is perceived as a lack. It is a lack to the extent of the gain that is the adjacent pile of earth 2 feet up (or '2 one foot-blocks', to forestall the objection that measuring, i.e. real - numbers, were being considered instead of integers). While it could be argued that all the teacher is doing is 'delivering' an image, I would counter that he is calling up an experience he is relying on the students to have had or to be able to imagine because of their memories of bodily actions. To be aware that we can never be certain what others are construing is one thing,

to deny that when the student steps into the hole s/he will fall is nonsensical; gravitational pull is part of reality.

Would Maddy accept this as ‘negative two perceived’? Certainly she would accept that the two-foot cubes removed from the hole were perceptible, the student could perceive an “impure set” (p156) with number property *two*. The question is how to interpret the hole! Is the hole just a two-foot negation or is it ‘where those two removed blocks go’? In the latter case, a notion of operation has been introduced (as it was indeed in ‘Stand and Deliver’, the teacher demands “Fill that hole, go on fill that hole: negative two, plus two is...” and a formerly unmotivated student becomes engaged and answers “zero”). Even in the former case, the idea of ‘level’ is necessary and Maddy does not claim to perceive the null set! So I do not think that it is possible to perceive negative two without employing an inverse operation. For this reason, Kitcher’s philosophy of mathematics (Kitcher 1984) may be more suitable to support this pedagogical representation, for he bases mathematical knowledge on primitive operations on material objects.

On (b) The ‘modern mathematics’ movement of the 1960s was a systematic and sincere attempt to answer question (b). The objective of the School Mathematics Project (SMP), founded in 1961, was “to devise radically new mathematics courses...which would reflect...the up-to-date nature and uses of mathematics” (SMP 1971, p i). The SMP designed their materials so that “in comparison with traditional texts, these texts pay more attention to an understanding of fundamental concepts” (ibid. p v). In particular, the notion of a set is seen as fundamental: “Chapter 2 introduces the basic idea of a set” (ibid. p vi). In the specific case of teaching negative numbers, the SMP chooses to separate the concepts of the mathematical object ‘ -1 ’ and the mathematical operation ‘ -1 ’ and distinguish between these notationally by the positioning of the negative sign. Their approach in Chapter 12 (ibid. pp 192 -208) starts in the same way as the ‘Stand and Deliver’ teacher: imagine a movement that does and undoes (in a linear manner), which is just the set theoretic concept of additive inverse “presented for instruction” (Shulman, 1987). I would expect Maddy to concur that this approach is more consonant with her theory than (a).

The problem of Maddy's set theoretic approach vis à vis learning is that it is rather formal, but the advantage is that there is a perceptual 'way in' to the formality. Comparatively, Resnik's ontology, (6.5.1), is perhaps more suited to a pattern-spotting approach, (like SMP 1985, p 57 - 62) and an anti-realist semanticist, such as Dummett, might emphasise the consistency of the meaning of statements involving directed numbers. Whether an experiential base for learning negative numbers is appropriate at all is disputed by both Fischbein and Freudenthal:

"the chapter of negative numbers has to be treated formally from the beginning. In [Freudenthal's] his view this is the first opportunity offered to a pupil to consider mathematical concepts from a formal deductive point of view." (Fischbein, 1987, p102).

The reason that Fischbein gives for this proposal is that:

"the concept of a negative number contradicted the concept of number itself as it had originally been developed in the history of mathematical reasoning. A negative number is a counter-intuitive concept because it apparently contradicts the notion of existence itself - if existence is concerned with practical meaning." (ibid. p97)

"the problem of $(-a) \times (-b)$ is much harder, ... because even very fine mathematical minds could not, for a very long time, completely rid themselves of the impact of implicit intuitive models ...[like] practical manipulations of concrete magnitudes" (ibid. p99)

While an educational theorist may advocate beginning negative numbers as an opportunity for introducing deductive reasoning, a practical problem for your average British secondary school teacher is that, given current curricula, nearly all pupils are to be taught about negative numbers, or rather, some models of negative numbers. It would be a hard task to teach the vast majority of 12-14 year olds formal mathematical deductive reasoning through the topic of negative numbers. Perhaps it is a worthy aim. A theory like

Maddy's might serve as a bridge from the perceptual directly to the formal without the need for taking temperatures or repaying loans.

6.9.3.2 *Some pupils work with negative numbers*

"a person may not be ontologically committed ... to what he appears to be committed to."
(Devitt, 1984, p43)

I chose this little quote to start this section on teaching negative numbers because it highlights a teacher dilemma: part of the teaching process is assessment and if mathematical knowledge is, in part, to be ontological commitment to, say, negative numbers, how can we assess this sort of knowledge? But, as I explained in chapter 2, I do not believe that I can project 'she has had a mathematical moment', 'he has ontological commitment', etc. onto people with whom I am acquainted because of something I have noticed about their behaviour. This is one of the reasons for my moving away from a psychologically inclined investigation. In particular, 'ontological commitment' is not an empirical concept. I cannot even theoretically devise an ontological commitment meter! So it does not make sense to search for evidence for this conceptual idea. But we say 'she's got it now', 'he's grasped it', 'the penny's dropped'. Attributions *are* made in practice. So with these caveats, I want to turn to school teaching experience and look at some work from an all-ability Y9 class which I taught for the Autumn term, 1996.

Provided in Appendix 9.2⁷¹, for a sample of pupils in the class, there are two items per pupil of work on negative numbers. The first is an A4 poster explaining something about negative numbers. To set this up, some discussion time was given in class, then the poster was completed at home. The second is the pupil's 'directed numbers mini assessment'. This was given, in class, as a silent individual task, a week or so later after the poster homework.

⁷¹ This work was not set for the purpose of presenting it to an academic audience, so teacher's comments and students' further corrections or attempts are all there. The 'better' posters ended up at an ATM conference!

What is my purpose in presenting some of the student's work? Because their own explanations and responses give those of us who are already 'committed' to negative numbers the opportunity to see what 'ontological commitment' *might* amount to or consist in. The examples from teaching experience help bridge the gap between the practical activity of teaching and the theoretical ideas under discussion. As a teacher I ask 'have you got it?', as a theorist I ask 'what does 'getting it' mean?' My answer to the latter question is that 'getting it' can be understood as 'ontological commitment'. And the discussion in the rest of this section indicates that this is not, itself, a straightforward, uncontested concept.

So a task for the reader is now to examine some pupils' work - with the notion of 'ontological commitment' in mind. For, although we cannot attribute ontological commitment, we can, and do, seek it! I have only made brief comments on their work, for I do not want to pre-empt the readers personal interpretation.

Both Lizzy and Anna work well with symbols, but collapse either addition or multiplication into the verbal 'and' making their 'explanation' invalid. Lizzy adds a moral tone: "2 rongs = 1 right"!

Bryony's advice on her poster is a sequence of images, including her own original item: the product of two negative numbers 'can't get any lower, so it must be back to positive'. Her assessment looks compatible with her advice.

Queenie has reproduced some of the class discussion on fell walking by Coniston and made her own North-South analogy. For Queenie, as Lizzy, 'negative' is 'wrong'! Her assessment shows that she does not feel free in using negative numbers themselves. She can subtract but avoids the use of negative number either as symbol, in the assessment, or as concept, on the poster.

Colin has an eclectic selection of tips on his poster. His assessment, (originally part C was incomplete), suggested a similar 'not put together'ness about his idea of negative numbers despite some symbolic competence.

Chapter 6: Mathematical objects

Siobohan used an idea from class discussion for her poster, and her assessment is also 'safe'.

From the theoretical point of view taken in this thesis, these data are presented to sharpen the concept of 'ontological commitment' - or lack of it! - not to claim that it is a construct capable of empirical verification.

6.10 Summary remarks

The aim of this chapter was to explain and justify the second part of the overall thesis on mathematical warrants, objects and actions. The main part of the chapter was concerned with analysing the concept of mathematical objects - the stuff maths is about. From an analytic philosopher's point of view, mathematical ontology has three principal, distinct channels: nominalistic, conceptualist and realist. These three aspects of theorising about what exists in mathematics are concepts in philosophical discourse which are still being discussed and the fuzzy boundaries between them redrawn. I think that the realist conception of mathematical existence the best out of the three, particularly if the issue of 'coming to know' is important, as in education. So in this chapter, I have said why I do not think so much of the nominalistic and conceptualist theories of mathematical existence, and then I presented some theories of mathematical existence for self-confessed contemporary realist philosophers of mathematics. The rationale for accepting the realist theory was strengthened by the historical analysis of 'ontological revolution' in mathematics. Given this case for realism in mathematics, I then sought to apply it in education. I did this by interpreting the philosophical term 'ontological commitment' in the context of learners and mathematical entities. The overall proposition is that there are objective mathematical entities, commitment to which is part of coming to know mathematics.

An irony of contemporary mathematics education is that the move away from formalism has lead to an embracing of the other anti-realist conception of mathematics, conceptualism. I hope to have shown here that a realist conception of mathematical objects is quite consistent with pragmatic teaching as well as academic analysis.

7. Chapter 7: Mathematical Action

All doing is knowing and all knowing is doing. (Maturana and Varela, 1992)

It is our embodied understanding that manifests our realist commitments. (Mark Johnson, 1987)

7.1 Introduction

Calculate: $\frac{57,000 \times 900}{380,000} =$

Assuming you responded to this stimulus, could you describe what you did? How would you characterise your action?

What I did was to firstly, 'knock the zeros off'. Secondly, I saw other factors and cancelled them. Finally, I tidied up so the answer was in whole number-fraction form. I would characterise my action as an automatic response. I am able to justify this action mathematically but, in practice, I just get on with it without any reflective thought; the notational form holds the mathematical structure. For me, doing this calculation exhibited 'automaticity'; whether and to what extent such action constitutes mathematical knowledge is the subject matter of this chapter.

The facility illustrated is not the same as a rote or, in Skemp's (1976) sense, a *purely* instrumental response. While non-reflectivity is a characteristic of this facility in action, i.e., as it is experienced, the facility holds, 'relational potential', (adapting Skemp's term which is contrary to 'instrumental'). The way I want to delve into the nature of this phenomenon of automatic mathematical action is to try to present a case for such action to be knowledge. Even if this claim might not be successful, the process of analysing this facility conceptually (epistemologically rather than psychologically), should provide a clearer guide to its nature. Some of the current debates on the British educational scene, in my opinion, are trying to address, what I have called 'automaticities', (for example, LMS, 1995, SCAA, 1997).

7.1.1 Examples of automaticity

Algebraic symbolic manipulation is an important part of higher school mathematics. Algebraic automaticity includes ability to solve simple equations as well as manipulation with more abstract mathematical objects. As an example of the latter, consider the process of expanding, or factoring, a finite binomial expression. More specifically, for those with fluent algebraic manipulation, $27x^3+27x^2+9x+1$ can automatically (and without awareness, necessarily), be factored as $(3x+1)^3$, and $(3x+1)^3$ can automatically be expanded to $27x^3+27x^2+9x+1$. The ‘relational potential’ of this facility may be expressed, for example, in terms of ‘seeing’ the coefficient of the x^3 as 3^3 , but the coefficient of the x^2 as 3×3^2 . I do not want to demand a person has the potential to give a formal explanation of their manipulative facility in order to be attributed with this sort of mathematical knowledge. If I did demand this, there would be no point in trying to argue for automaticity as knowledge, for such reasoning would be already subsumed under standard mathematical knowledge claims. Another potential capacity for someone with ‘binomial automaticity’, would be the capability to regard $8y^3+8y^2+4y+1$ as not a representation of the cube of $(2y+1)$, as a ‘pattern spotter’ might conjecture. The reasons given could be very scant: ‘there are no 3s’, say. The point is that the binomial theorem has a ‘shape’ or ‘rhythm’ and that knowledge of that theorem includes the mathematical actions of expansion and factoring and a ‘smell’ for that which is wrong as well as correct.

One of the points about automaticity I want to communicate is its natural, almost instinctive, feel. Here is an example: recently ‘old money’ came up in a family conversation. Our new money, new technology child wondered how we calculated with those awkward units. For example:

$$\begin{array}{r}
 \begin{array}{rrr}
 \pounds & s & d \\
 8 & 13 & 11 \\
 \times & & 7 \\
 \hline
 60 & 17 & 5 \\
 4 & 6 &
 \end{array}
 \end{array}$$

The currency has been decimalised for 27 years and I have not been doing these sort of calculations during this period! Yet I did not (just) know how to do this calculation with my head - for I am able to reason out how to work in a given number base - but I knew it 'in the body'. Anyone with a knowledge of different base arithmetic could perform this task, given the base-information, but I doubt if they would have £.s.d. automaticity.

Without some automaticities a student's progress can be restricted or made laborious. An example of this was reported to me by a university mathematics lecturer, (Bill Cox, personal comm.). Of his first year mathematics undergraduate students, a substantial proportion did not answer $\int \frac{dx}{\cos^2 x + \sin^2 x}$ by 'seeing' the denominator as 1. Instead they used standard substitution techniques, like letting $t = \tan \frac{x}{2}$, which eventually yielded the correct answer but with excessive effort.

How can automaticities be recognised? Clearly not by observation of behaviour alone; a very fluent operator may just stare into space doing algebraic manipulations in her head! A first step is to catch the process in oneself. It is difficult to do this because the attempt to observe ruins the automatic, instinct-like, character⁷². Nevertheless, some automaticities may be self-observed if one carries out the following little task:

(i) Let $y = x^2 - 13x + 42$, what are the roots of $y = 0$?

⁷² If I become aware of uttering something fluently in a foreign language, this very awareness usually trips up my fluency and my next utterance is laboured.

Chapter 7: Mathematical Action

- (ii) Let $y = x^2 - 14x + 42$, what are the roots of $y = 0$.
- (iii) Make up a quadratic with two positive roots, subject to the condition that the vertex of its graph is at $x = 6$.
- (iv) Can you make one that crosses the y -axis at $(0,42)$?

Part of this task will be discussed in 7.3.3.

As a grammatical aside, ‘automaticity’ is used both in adverbial phrases and as a noun. When I say ‘I did the calculation *with automaticity*’, I mean that I did the calculation in a fluent, non-rote, curtailed manner. The phrase has the same structure as ‘she danced *with spirit*’ (i.e. ‘spiritedly’). This is to be distinguished from ‘I did the calculation automatically’, i.e. by rote. I have also used the term ‘automaticity’ in phrases like ‘executing an automaticity’. In this phrase, the referent of the noun ‘automaticity’ is an action. For example, the action of solving an equation in a fluent, non-rote, curtailed way is an ‘automaticity’.

7.1.2 Piaget’s legacy

The idea that action is intrinsic to knowledge is a central theme of Piaget’s work. Piaget’s influence on the field of education has been, and continues to be enormous: he wrote over forty books spanning more than four decades on biology, psychology, knowledge, intelligence, and how these are integrated and structured as a child develops and matures. Piaget called himself a ‘genetic epistemologist’: one who studies how knowledge grows. Child development includes knowledge development and “the goal of genetic epistemology is to link the validity of the knowledge to the model of its construction” (Vuyk, 1981 p26). So Piaget’s project has a philosophical dimension. His biologist’s roots fixed him to the physical foundation of humans’ sensory mechanisms. Suppose these mechanisms were fundamentally different then, Piaget asserts: “our fundamental concepts would be turned upside down, not just because of the way things appeared to us, but because of our means of action” (Piaget, 1971a, p 271). Piaget invented some quasi-mathematical structures to describe stages of knowledge development: mathematics was

used as a scientific model. My aim, here, is to make a different link between mathematics and action: I argue that some actions are intrinsic to mathematical objects: ontological commitment to the mathematical entity is a consequence of mathematical activity with it, as the binomial example above illustrates.

A thesis could be written on Piaget's relationship between empiricism and rationalism (but not this one!): Boden (1979) paraphrases Piaget as saying "Empiricism describes the growth of knowledge in terms of genesis without structure, whereas rationalism offers us structuralism without genesis" (p 88), but Piaget's later study of dialectics¹ attempted to interweave the two (Vuyk op. cit. p17). This dialectical approach is compatible with Piaget's attraction to the Kantian notion of 'synthetic-a priori', (Boden, op. cit. p91) and it is related to the vexed question of how (a) we know necessities and (b) how necessities are related to the contingent world. I address these issues by focusing on a small part of mathematical knowledge. I am not following Piaget's work, yet recognise the foundation his theories provide in linking action to knowledge.

7.1.3 This chapter's proposition

The gist of this chapter is that automatically executed actions with mathematical entities can constitute knowledge by virtue of their (warranting) form. This is due to mathematical objects and mathematical forms (like some algorithms) being inextricably linked. This is part III of the overall thesis. Furthermore, as educational corollary, (iii), individuals require some procedure-embodiment for mathematical knowledge, i.e. learning mathematics involves developing a capability to execute some mathematical procedures automatically.

I want to make a case for this 'automaticity' to count as mathematical knowledge, and hence for teaching mathematics to include facilitating students' automaticities as a general aim. As Tall and Thomas remark: "one of the best indications of understanding

¹ "Piaget's meaning of the concept differs from Hegel's thesis-antithesis-synthesis" (Vuyk, 1981, p18).

Chapter 7: Mathematical Action

the ability to sense that something is true in an immediate manner, without recourse to a formal proof." (1991, p49). These 'automaticities' are a form of non-propositional knowledge. The relationship between this type of knowledge and propositional knowledge (which I have focused on in previous chapters) is based on the notion of 'intention' and the potential variability of intention with respect to a given mathematical action.

7.1.4 Outline of the chapter

The phenomenon of automaticity has been recognised by authors from various different traditions. In the following section, 7.2, representatives from this extensive literature are presented. In 7.3 the thesis is elaborated, an argument presented and some objections met. Finally, 7.4 rounds off by examining implications for mathematics in education, in particular I point out some implications regarding the use of computer algebra systems (CAS) in teaching.

7.2 Automaticity from different perspectives

I have used the term 'automaticity'⁷⁴, but there are other terms which seem to capture the same phenomenon: For example, the terms that teachers might use to express a student's capacity for fluent calculation or interpretation include phrases like: 'get it off pat', 'it needs to be hard-wired'. This reflects the idea that the concept of cosine, say, is 'part of the brain' and, given a right angle triangle calculation, the correct ratio is selected without undo reflection.

Another term, offered in the specific context of mathematics education by Krutetskii, is that of 'curtailment'. The ability to *curtail* mathematical reasoning is one of Krutetskii's

⁷⁴ The term 'automaticity' was suggested to me by Eric Love.

characterisations of the 'capable' pupil in mathematics⁷⁵, (Krutetskii, 1976, p264, p350), which he describes in the following terms:

“curtailed conclusions also occupy a definite place in problem solving, when the pupil is not aware of the rule or the general proposition by which he is actually operating. As a result, in solving a problem he does not perform the whole chain of reasons and deductions that form the complete, detailed structure of the solution.”
(ibid. p 264)

The term 'compression', used by Thurston (1995) and then by Barnard (1996), also refers to the same phenomenon as that of curtailment used by Krutetskii. Barnard draws attention to the efficiency of what he calls 'compressed entities' in manipulation, but considers them possible to unpack "whenever needed". Dubinsky's notion of 'encapsulation' is also similar (Dubinsky, 1991).

The metaphor of 'fluent', favourite of linguists, has often been adapted to describe a student's mathematical performance, as David Pimm discusses (Pimm 1995, particularly pp 170 - 183). This notion of 'fluent' is close in meaning to 'automaticity' - fluency requires some automation: "Currently, when working on algebraic forms, I am encouraged to suppress 'meaning' in order to automate and become an efficient symbol manipulator" (Pimm, op. cit. p 108). This idea that procedural efficiency requires a 'just do it' attitude to routine mathematical calculations, which can be explained if required, underlies the proposition which I am asserting:

'automatically executed actions with mathematical entities constitute knowledge, per se, by virtue of their (warranting) form, and that learning mathematics involves developing a capability to execute some mathematical procedures automatically'.

⁷⁵ Other features of his characterisation of the capable include being able to generalise, having flexible mental processes, being able to reverse reasoning and being able to hypothesise.

Chapter 7: Mathematical Action

Another word which suggests a connection between action and thought is 'embodiment'. The way in which I shall use the term follows Mark Johnson. In the preface to his 1987 book he says (of the book) "*The Body in the Mind* is thus an exploration into some of the more important embodied imaginative structures of human understanding that make up our network of meanings and give rise to patterns of inference and reflection at all levels of abstraction." (p xvi). This idea, that abstraction starts with bodily action, is part of the thesis here.

In a similar vein, John O'Neill's discussion of 'Cognition and Action' constitutes a philosophical argument for the bodily basis of meaning, (O'Neill, 1984, pp35-50). The key point of this argument is to privilege action over perception, which is relevant to the present topic of automaticity. His term 'basic action', following Davidson, is characterised as being an action, (possibly incorporating an object), which is executed with minimum of intentionality and, though not specifically mathematical, has a sense akin to 'automaticity'. I shall discuss this further below in 7.2.3. O'Neill argues that the behaviourist and representationalist views⁷⁶ of mind fail to account for the variability of intentionality which mental states can have. He then sketches an 'actionist' theory of mind wherein:

"our ability to make representations, to use one object to refer to another, is founded on our direct access to objects in the external world given by action on them. The content of our representations is founded on our non-representational knowledge of the success or failure of the actions they guide." (O'Neill, op. cit., p309)

The point is that in order to make representations - sounds, diagrams, symbols, for example - some other, hence non-representational, facility is employed: a 'basic action'. The distinction between 'basic' and 'non-basic' actions is crucial in O'Neill's argument that actions can be non-representational. Basic actions are those which "one can do without doing anything else...Non-basic actions cannot be known directly" (p 317-8).

⁷⁶ Both of which he claims are 'formalist' accounts and his thesis is entitled 'Against Formalism'.

'Automaticity', as I have been trying to characterise it, is not the same as Vergnaud's 'théorème en acte' (1981) because, for Vergnaud, reflection is not available:

"Le concept de 'théorème en acte' désigne les propriétés des relations saisies et utilisées par le subject en situation de solution de problème, étant entendu que cela ne signifie pas qu'il est pour autant capable de les expliciter ou de les justifier."⁷⁷ (p10-11)

The subject exhibiting automaticity, unlike Vergnaud's subject, would be 'capable of explaining or justifying' the connections which he has 'grasped and used'.

To recap: the term I have chosen to use to express a 'hard wired' mathematical capacity is automaticity. It is suitable because it describes action that can be executed without necessary reflection, and has the connotation that an embodied relational understanding, ('relational' implies that the subject has some capacity to unpick that understanding), is part of the individual's mathematical practice. Whether, or in what sense, 'automaticity' counts as mathematical knowledge is going to be a more difficult case to make and is the focus of section 7.3. In the following subsections, I aim to put flesh on the meaning of 'automaticity' by illustrating that the phenomenon - albeit named differently - has been recognised from psychological, linguistic, mathematical and philosophical perspectives.

7.2.1 *Krutetskii*

Of a 'capable pupil': "he proved the remaining algebraic theorems freely, without reflection" (p246)

Krutetskii's seminal work on mathematical abilities in schoolchildren gives many specific examples of pupils' facility of automaticity, (particularly within his Chapter 13 'Characteristics of Information Processing' pp 237-294). This facility is referred to by

⁷⁷The idea of the 'enacted theorem' points out the properties of the connections grasped and used by the subject while trying to solve a problem, being granted that it does not mean that he is, for all that, capable of explaining or justifying these connections.

Chapter 7: Mathematical Action

Krutetskii as the ability to 'curtail', as quoted above. Krutetskii's psychological study was seeking to characterise mathematical ability and proceeded to seek attributes of 'capable' pupils which were not present in those whom he termed 'average' and 'incapable'. One such attribute was this capability to work as Krutetskii's student O. V. did when asked "to solve: $(C+D+E)(E+C+D)$ " (p241), not having worked 'trinomials' before. This 'capable pupil' first 'sees' the given product as a square, then represents it as a binomial, with which s/he is familiar. The pupil expresses this action as: " $(C+[D+E])^2$...[for] as soon as I combined D and E into one term, I got a binomial. ... A 'term' can be any expression". Krutetskii sums up this achievement as: "the pupil has composed an algorithm for solving all problems of this type" (p 241). The pupil's action (bracketing, use of binomial) is a form, or algorithm; this connection between behaviour (the action) and logic (the algorithm) is, I claim, an aspect of mathematical knowledge.

Krutetskii's report on his psychological project is a mine of information and detailed examples, such as the one given above. Krutetskii's report supplies empirical evidence of the capability of automaticity by presenting it in specific cases. He also gives non-examples, (e.g., p242, p244, p246) which helps further clarify the capability. These non-examples, show how, without this capability, a pupil's flow stops, how structures and generalities are not perceived and how notation is interpreted without sensitivity to its symbolic function. Krutetskii makes it clear that the 'curtailment' he exemplifies from his data is not the same as "unreasoned and unmotivated" omissions, (as apparent in some 'incapable students' reasoning) (p267). For he reports that at the researchers' request the 'capable' pupils "expanded curtailed structures to their full structure" (p270). The ability to operate without reflection, yet be able to fill in the gaps and give the rationale for certain moves on request, distinguishes 'automaticity' or 'curtailment' from a 'théorème en acte', in Vergnaud's sense. I do not want to suggest that, for example, a correct but, to the pupil, ungeneralised and, as yet, ungeneralisable, use of the distributive law constitutes knowledge of these multiplicative structures. More specifically, though Jimmy can tell you what 7×13 is by working out $7 \times 10 + 7 \times 3$, à la *théorème en acte*, this does not

mean that he can describe this process abstractly⁷⁸ nor use the distributive law's structure to support calculations at the limit of his mental range, as 23×47 might be. What I am trying to draw attention to is that the pre-verbal *théorème en acte*, is different from a-verbal proficient action, which could well have been given the same name: 'enacted theorem'. Binns and Mason help clarify this distinction:

"Students may behave as though they know these laws [conservation of equality and distribution] but may in fact be quite unaware they are employing any general principle" (Binns (now Bills) and Mason 1993, p 3)

This phenomenon of being unaware of employing (what is in fact) a general principle is different from employing, without awareness, a general principle, the *generality* of which is understood. Krutetskii claims that in the case of 'capable' pupils they do grasp the generality, albeit in a non-explicit, curtailed manner, but the 'average' or 'incapable' pupils require "preliminary, gradual generalization on the basis of practice" (Krutetskii, op. cit. p 246). This practice gives the opportunity for a stimulus-response capacity to be established of the *théorème en acte* type.

Krutetskii's work was on high attaining pupils, but this investigation is into the nature of higher school mathematical knowledge generally. Nevertheless Krutetskii's observations are relevant. Firstly, they provide evidence for the distinction between curtailment and *théorème en action*. Secondly, they point to the a-verbal aspect of these algebraic facilities - because they are so spontaneous - which is invariably lost when a pupil has to have been didactically instructed into the same algebraic actions. These capable pupils seem to have been born with some hardwired mathematical capacities - i.e. with 'automaticities'. The question I want to address is: in what sense does this capability constitute knowledge?

⁷⁸ Any sort of abstract generalisation could be acceptable, not just ' $a(b+c)=ab+ac$ '.

7.2.2 Mark Johnson

Evidence of ‘automaticity’ from Mark Johnson’s book ‘The Body in the Mind’ is quite different from that evidenced by Krutetskii’s work. Johnson is not concerned specifically with mathematics, although he does consider logic. His aim is to give an account of meaning that incorporates action-cognition. His account of logical, deductive knowing through bodily experience is particularly relevant to a discussion of the knowledge status of mathematical automaticities.

Johnson’s work, inspired by Kant, locates *imagination* as the prime mover in developing human understanding. Image schemata⁷⁹ which are by nature abstract, are the foundations for metaphorical imaginative leaps of meaning-grasp and meaning-creation. These image schemata are structures that have a basis in physical experience. Johnson exemplifies this with image schemata like verticality (p xiv), containment (p 21, p39) and force (p 42). Ups and downs, ins and outs, and pushes and pulls, are bodily experiences; they are “embodied” (p xx). Johnson’s key point is that our understanding of, say, the complex concept of measure in applications like “prices keep going up” (pxv), relies on the verticality schema⁸⁰.

Because my particular interest is in mathematical reasoning, which includes logical inference, I quote, at length, Johnson’s explanation of how knowledge of some simple logic is achieved:

“it follows from the nature of the container schema (which marks off a bounded mental space) that something is either *in* or *out* of the container in typical cases. And if we understand categories metaphorically as containers...then we have the claim that everything is either P (in the category-container) or not-P (outside the

⁷⁹ “Image schemata are abstract patterns in our experience and understanding that are not propositional in any of the standard senses of the term, and yet they are central to meaning and the inferences we make” (p2)

⁸⁰ Johnson cites this as an example of ‘quantity’.

container). In logic this is known as the ‘law of the excluded middle’...the principle ‘Either P or not-P’ has an intuitive basis in our daily experience with containment.

A second logical relation that is experientially motivated by containment is transitivity ... if a set A is a member of (is contained by) set B and set B is a member of (is contained by) set C, then A is a member of set C.⁸¹ ...

This basic logic of containment also motivates the equivalence for double negation, $\sim \sim P \equiv P$

such inferential patterns arise from our bodily experience of containment” (p 39-40)

The reader may recall that, in 5.4.3, I suggested that modus ponens reasoning, ‘ $P \Rightarrow Q$ & P then Q’, was also attributable to bodily experience of containment. For, by containment, if P is ‘inside’ Q (à la image of a Venn diagram), then whenever P occurs, Q inevitably occurs too.

The power of Johnson’s work has been recognised in the mathematics education field. Presmeg (1992b) uses the linguistic aspects of Johnson’s theory to explain how metaphoric and metonymic associations are, psychologically, both essential for the learning of new mathematics. They depend on the subject’s imaginative facility. But it is Johnson’s explanation of the intrinsic importance of bodily experience in developing meaning which is relevant to my claim of that mathematical action can be knowledge. This aspect of Johnson’s work is also noted by Presmeg in her review of his book (1992a): “an important part of Johnson’s book ... is reflected in its title.. internalisation of bodily action is a central aspect in the Piagetian formulation as in Johnson’s”, but is not central to her use of his work.

⁸¹ This does not seem correct, set theoretically: for, if $A \in B$ and $B \in C$ it does not follow that $A \in C$. For example, if $A = \{a\}$ and $B = \{\{a\}, b\} = \{A, b\}$, then $A \in B$, and if $C = \{B, c\}$, then $B \in C$, but $A \notin C$.

But if $A \subset B$ and $B \subset C$ then $A \subset C$. This containment is illustrated by $A = \{a\}$ and $B = \{a, b\}$, so $A \subset B$, and if $C = \{a, b, c\}$, then $B \subset C$ and $A \subset C$.

7.2.3 John O'Neill

Davidson's notion of 'basic action' has been used by John O'Neill in his action-based theory of meaning (O'Neill op. cit.). O'Neill declares that mental states are "intrinsically intentional" (p311), but this does not imply that attention is paid to this intrinsic intentionality at the moment of activity. This is a familiar situation in mathematical activity; I just calculate. In the words of the poet Basho "how unaware, how useful".

O'Neill argues⁸² that "'moving my body in just the way required to tie my shoe laces' is not a re-description of 'tying my shoe laces'" (p 321). The distinction is made by associating the former with the novice and the latter with the expert. The novice attends to his/her fingers; the expert performs shoe lace tying as a basic action: "in tying one's shoe laces the fingers move, just as in raising one's arm the muscles flex." (p 322) The complexity of such basic actions can increase: O'Neill exemplifies this through spade-action and blind-stick-action. These actions become *basic* when the intentionality in executing them has died away. Thus the blind man's stick is an extension of his body, "if the ground falls away sharply, [he] knows this directly" (p 323) rather than "infers" or "represents" that state of affairs. Negligible intentionality in use of his stick does not imply that the blind person does not know that he is holding it. O'Neill's analysis does not go into the educationally relevant epistemological problem of how an action becomes 'basic'.

Within mathematics education, Dave Hewitt has devised some practical teaching strategies which aim to minimise intentionality with respect to a mathematical process by diverting attention, (reported in Pimm, 1995, pp95-6) The aim being to encourage the development of students' algebraic or arithmetic automaticities.

⁸² against Davidson, (O'Neill, op. cit. p321)

7.2.4 Wang Hao

Wang is a philosopher and a mathematical logician whose aim, (in particular in his 1974 collection from which reference is taken), is to investigate “the relations between logic and knowledge” (p20). Wang draws himself away from the highly abstract world of formal mathematical logic to investigate how logic can help understand human knowledge generally. Because he does not want his study to become detached, as often logical treatises are, Wang “look[s] at mathematics in terms of the practice and the activity from several angles” (p21), i.e., Wang shows his interest in epistemological detail by his use of specific examples (drawn from mathematics, as that is his expertise). For example, in the following quote, Wang unpicks mathematical activity associated with a particular well known problem:

"The mind participates actively in seeing, e.g., an array of numbers⁸³ as paired off suitably to create a new uniformity. This 'seeing as' enables us to take in at a glance the 5000 pairs of numbers which all have the same sum, 10 001. In this respect, the dots are not 'mere abbreviation' either, because they, or something else like them, are indispensable for grasping the array of numbers at one go; they embody the formal fact that we see the 5000 pairs as a whole string with a definite beginning, a definite end, and a definite way of continuation. In doing this calculation, one is likely to make (mental) experiments such as trying to look for suggestions from summing a small number of integers. But calculation is not itself an experiment, since once the path is found, certainty intervenes." (ibid. p 131)

⁸³ Wang is referring to the problem of finding the sum of the first 10,000 integers by Gauss' method

1	2	5000
10,000	9,999	5001

This passage illustrates Wang's notion of 'mathematical perception' as "the mind participating actively in seeing" which is done automatically, "at a glance". This is a good way of expressing the notion of automaticity that is central to this chapter. In talking about the role of the tabulation and the dots, Wang observes that cognizing beings like me need a supportive notation to symbolise the mathematical relations. What is more, Wang goes on to tie in the notion of "certainty" with "the path of the calculation" which, in other words, is the mathematical warrant or form.

7.2.5 Wittgenstein

Wittgenstein does not present an action-based theory, as Mark Johnson's, nor a mathematically orientated one like Krutetskii's or Wang's. As is well known, Wittgenstein's later work centres around language and meaning, though he draws on ordinary experience of basic mathematics and daily life to provoke and to test his conjectures about meaning. The question of logical inference, "the hardness of the logical *must*" (Kenny, 1994, p233), clearly interests him; discussions on the sense of logical compulsion are to be found in (at least) Kenny, 1994, 'The Blue and Brown Books', (Wittgenstein, 1972) and his 1939 lectures on mathematics, (Wittgenstein, 1976). In these works Wittgenstein writes on necessity and logic and how these notions relate to both mathematics and general experience.

This passage illustrates Wittgenstein's concern with mathematical meaning, necessity and experience:

"the point is that the proposition ' $25 \times 25 = 625$ ' may be true in two senses. If I calculate a weight with it, I can use it in two different ways. First, when used as a prediction of what something will weigh - in this case it may be true or false, and is an experimental proposition. I will call it wrong if the object in question is not found to weigh 625 when put in the balance. In another sense, the proposition is correct if calculation shows this - if it can be proved- if multiplication of 25 by 25 gives 625 according to certain rules. It may be correct in one way and incorrect in the other, and vice versa. It is of course in the second way that we ordinarily use

statement that $25 \times 25 = 625$. We make its correctness independent of experience. In one sense it is independent of experience, in one sense not. Independent of experience because nothing which happens will ever make us call it false or give it up. Dependent on experience because you wouldn't use this calculation if things were different. The proof of it is only called a proof because it gives results which are useful in experience" (Wittgenstein, 1976, p41-2)

This passage exemplifies Wittgenstein's recognition of the physical reality of elementary arithmetical propositions, as well as their formal truth. But he keeps these two notions apart. I want to make a case for connecting these notions via an action, or bodily, based theory of meaning.

While Wittgenstein never concedes reference outside of a language game, he does examine the curious force of logical necessity (which appears to refer to a non-linguistic reality): "...that 'what seems to be a logical compulsion is in reality only a psychological one' - only here the question arose: am I acquainted with both kinds of compulsion then?!" (Kenny, op. cit., p 231). His problem is to account for where this sense of necessity comes from: "how do we become convinced of a logical law? ... We might say: It is some primitive kind of experience which corroborates logical laws ... [but] the laws of logic are not corroborated or invalidated by experience." (Wittgenstein, 1976, p199-200). Although I agree with this statement, its very assertion illustrates my divergence from Wittgenstein's point of view, which is that certain kinds of experience *are* instantiations of logical laws.

One of the metaphors Wittgenstein explores when discussing logical necessity is 'mechanical'. And this concept relates to 'automaticity'. I shall argue for a sense of 'mechanical' in fluent mathematical reasoning which is more physical than Wittgenstein's metaphorical associations. In his 1939 lectures, Wittgenstein discusses 'logical machinery' (pp196-200) and shies away from its real existence (p198); it remains a metaphor for him - the examples he musters are but empirical and contingent or social-linguistic and not of 'logical type' at all. In another work, (Kenny, op. cit.) the notion of 'machine as symbol' gives a slightly different conception: "the movement of the machine-

as-symbol is predetermined in a different sense from that in which the movement of any given machine is predetermined" (p234). At first sight, a link seems to be made here between the material and the linguistic. But the 'machine-as-symbol' is also a mere model without a material reality, albeit that we only understand its meaning because of our experience with pulleys and gears and so forth. 'Automaticity' has a sense of an actualised 'machine-as-symbol', even though this is not within Wittgenstein's conception.

Part of the point of reading Wittgenstein is the philosophical work one has to do to make an interpretation of his aphorisms and paradoxes; he is a task based writer. When Wittgenstein says: "...I am to act and *not* consider." (Kenny, p242), he echoes evidence already discussed in this chapter, another example of which is the Krutetskii pupil, who "To the experimenter's question 'How did you solve it?' replied: 'Here there is nothing to think about - just look at the example and write'." (Krutetskii, op. cit. p266).

This brief collection of examples from Wittgenstein's writings show that he was concerned about the problems of coming to grasp ideas of logic and mathematics. He also recognised phenomena which included actual calculating and deducing as indicating these problems. However, his linguistic analysis did not allow an experiential answer as he always divided the language game from the experience.

7.3 A case for automaticity as knowledge

In this section I shall try to make a case for 'automaticity as knowledge' and answer some objections to this assertion.

The examples, tasks and perspectives illustrated in the previous section should give credence to the existence of the phenomenon of automaticity. The question which I turn to now is whether, and in what circumstances, automaticity may count as knowledge. There are crucial and subtle distinctions, as I indicated in 7.2, between habitual, routine and algorithmic actions with mathematical entities and 'automaticities' which are candidates for knowledge. For example, the routinised action of the pupil who obediently divides fractions by 'turning the second one up side down then multiplying them' is the

very anathema to the mathematical knowledge which 'automaticity' connotes. Yet the observed behaviour of the pupil 'going through the motions' of, say, dividing one fraction by another, and the one whose behaviour was automatic, curtailed or compressed, may well appear very similar. This is one of the reasons why observation, per se, has its limitations when trying to analyse this phenomenon. For knowledge is not to be inferred by such or such a behaviour. An argument for automaticity as knowledge cannot come from analysis of behaviour, the use of examples of behaviour are given to help the reader tune into the meaning of the term.

Unlike Giaquinto's visualisation (see 5.4.5), automaticity - mathematical knowledge in action - may often be a more gradual process than the 'ah-ha! 'seeing' of a geometric truth. But, as Krutetskii's pupils' curtailed reasoning exemplifies, those with a 'mathematical cast of mind' are able, sometimes, to curtail their reasoning in quite novel situations. Automaticity, nevertheless, seems more applicable to algebraic, arithmetic or logical reasoning, than geometric, problem solving or modelling mathematical activity, and I shall focus my discussion on the former.

A feature of automaticity is that new (for the individual) propositional knowledge may arise from such an automatic mathematical action which I am claiming is a sort of non-propositional knowledge. Each 'move' in the symbolic deduction is executed without specific awareness or reflection, but the result is novel. For example, in 3.2.3.1 on the ellipse-like locus, the eccentricity of the ellipse was a novel result, to me, as a consequence of the preceding algebraic manipulations.

With automaticity, unlike visualisation, it is the process itself which is the knowledge, (even though something else new might be 'seen' as a result of this process). This is because, as I shall elaborate on below, the process is a mathematical form (structure or argument) which is itself a mathematical warrant.

So a working definition of an 'automaticity' may be taken as: 'an execution of an algorithm or logical deduction together with a potential rationale for that algorithm or deduction'. This should give a flavour for the sort of action/process/grasp I am trying to

capture. However, this particular attempt to characterise ‘automaticity’ leaves as problematic the ‘potential rationale’. I relate this to the philosophical notion of ‘intention’ in section 7.3.2.

7.3.1 Automaticity: an example

As the examples above illustrate, the ‘potential rationale’ may not be the driving force for the calculation or algebraic reasoning in question, but subliminal. In order to make the claim that ‘automaticity’ is mathematical action-knowledge, a specific mathematical process is examined. By analysing the connection between the action required for the calculation, i.e the form of the process, and the warrant for the truth of the resulting proposition, I want to show that these can be structurally the same, thus showing how action can be interpreted as mathematical knowledge.

The process I have chosen to analyse is two digit by two digit Vedic multiplication⁸⁴. This is used because (i) Vedic multiplication is traditionally taught as a process based on the Sanskrit aphorism ‘crossways and vertical’. This aphorism captures the dynamic physical representation of the mathematical process involved in multiplying; (ii) the process has a culturally independent, structural, ‘British-schoolgirl’ explanation and this explanation serves as mathematical warrant.

The Vedic method for multiplying pairs of two digit numbers is illustrated by an example:
87 multiplied by 96:

8	7	1	3
9	6		4
<hr/>			
8	3	5	2

•

⁸⁴I learnt this method of multiplying at HIMED, February, 1997.

And the answer to the multiplication is **8,352**⁸⁵.

In Vedic multiplication, unlike the British standard 'long multiplication', blind adherence to the algorithmic instructions given will not always produce the desired correct answer.

At this juncture, the reader may like to construct an example that follows the Vedic algorithm, described in the footnote, yet does not get the correct answer (without making further adjustments).

Using this Vedic method for calculating involves understanding the form, i.e. the structures involved, as well as the 'moves', what to do. Without the understanding of the form, (part of which is the base 10 representation of whole numbers), mistakes are more likely, for reasons the reader may have found in constructing a non-successful manipulation. And it is this deeper sense of form that is required for this automatic use of such an algorithm. The *form* of the calculation *is* the processes's validity and it is the mathematical warrant for the calculation.

What is the 'form' of Vedic multiplication? Why is this form a mathematical warrant?

Before drawing attention to 'Vedic form' and making a case for it to be a warrant, I want to clarify what knowledge claims are being made. There are two obvious contenders: (a) I know $87 \times 96 = 8352$; and (b) I know that calculating 87×96 using the Vedic method will yield an arithmetic truth. The claim I want to make is (b). For fallible human calculation, as in (a), can always be in error in particular cases, but with general logic-based methods there is a possibility of (local) completeness and consistency, i.e., knowledge.

Vedic multiplication form:

⁸⁵ This is achieved as follows: 87 and 94 are each subtracted from 100 and the product of these 'discrepancies' is written as the right side of the answer (52, in this case). Next, subtract the discrepancy of either one of the original numbers from the other (resulting in 83 in this case), this number is written as the left hand side of the answer. Setting the working out as indicated above ensures that the answer, in standard notation, of eight thousand three hundred and fifty two is correct.

Chapter 7: Mathematical Action

1 The numbers are base 10, i.e. Th H T U *Write the two numbers in base 10.*
 numbers

2 The multiplication 87×96 means 'how *Set out* $\begin{array}{r} 87 \\ 96 \end{array}$
 many is 87 lots of 96? (or vice versa)'

3 Define 'discrepancy' - $d(n)$ - as the *Find the two discrepancies, place*
 difference between the order of *them thus:* $\begin{array}{r} 87 \\ 96 \end{array} \quad \begin{array}{r} 13 \\ 4 \end{array}$
 magnitude of n , and n , (relative to base
 10, here). For example, $d(87)=100-$
 $87=13$, as 100 is the order of magnitude
 of 87.

4 The answer to the multiplication is given *Write these two numbers adjacent:*
 by $100-d(87)-d(96)$ in the H place and *83 52 to get the answer*
 $d(87) \times d(96)$ in the U place.

5 **This both means and is because: ***

$(100-d(87)-d(96)) \times 100 + d(87) \times (96)$ = *The answer is 83 hundreds and 52*
 $(100-d(87)) \times (100-d(96)) = 87 \times 96.$ *units or*

8 thousand 3 hundred 5 tens and 2
units.

This sequence 1 - 5 both justifies the Vedic multiplication algorithm⁸⁶ and explicitly specifies the actions involved in using it to do multiplications. It also explains why 64×97 is not 61108; understanding of place value, not just the algorithm is needed to justify why 6208, not 61108, is 64×97 .

7.3.1.1 *Vedic form as mathematical warrant*

To show that this ‘crossways and vertical’ method, can be a mathematical warrant, it is enough to refer back to the structure *. However, the crucial step in mathematical knowledge development is the ‘internalising’ of this method *as* warrant. One way to understand what this might mean is to envisage two users of the method one of whom can demonstrate a structural analysis of the method and the other who operates instrumentally. (These approaches mirror the regular and italic text above). The idea is that warrant must contain a potential for increasing detail or explanation of the form, yet in executing the process given by that form the action may be highly curtailed.

To recap: in this sense of automaticity, I can make a knowledge claim: ‘I know that calculating 87×96 using the Vedic method will yield an arithmetic truth’, when (i) I can execute a calculation of that type without substantial reflection; and (ii) I am able to justify my moves with the symbols I use via a mathematical warrant; and (iii) the form, or structure, of the moves is the warrant used to make the justification (iii). There are several objections to this sense of automaticity being knowledge some of which shall be answered below in 7.3.4.

7.3.2 *Automaticity as non-propositional knowledge*

The claim here is that ‘automaticity’ is non-propositional knowledge. That is, it is knowledge of the tying shoe-laces type but with mathematical objects as ‘laces and bow’ and mathematical warrants as ‘expert motion of fingers’. The idea that an action can be

⁸⁶ So $d(105) = -5$ and the Vedic algorithm, with adjustments, still works: $105 \times 87 = 9200 - 65 = 9135$

Chapter 7: Mathematical Action

knowledge, is different from the situation discussed earlier, particularly in chapters 5 and 6, where knowledge of mathematical propositions, and the referents of those propositions, were the main issues.

At this juncture it may be helpful to compare propositional and non-propositional knowledge about the same item of higher school mathematics. A student may be expected to 'know' the following proposition: ' $x \in R, n \in Z, f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$ '. Of course, this proposition may be expressed in a more discursive form or with different notations. The corresponding non-propositional action-knowledge could be described as 'knowing how to get the derived function of a monomial' together with a potential to warrant. Potential justifications could be: a formal derivation of the derivative using limits, or a quasi-empirical estimate of the smoothing function of differentiation, (visualisable for small $n > 0$), together with a falsification of the 'take the exponent and subtract 1' rote procedure in the case of other classes of functions like $g(x) = e^{nx}$. (The strength of the person's action-knowledge is, perhaps, related to the warrants available for justification and the flexibility with which these are employed.)

The argument for automaticity being knowledge rests on the following:

1. There is an analogy between automatic actions with physical objects and automatic actions with mathematical objects. This was shown in 7.2.
2. These automatic actions with mathematical objects (the ones which I want to claim have this knowledge status) have a structure which can be associated with a mathematical justification. This was explained through analysis of the Vedic multiplication example.
3. It is possible to distinguish theoretically between rote-action and knowledge-in-action.

I now turn to 3. of this list and make the distinction between knowledge-action - 'automaticity' - and habitual or rote-action, by going back to some of the 'task' suggested

at the end of section 7.1.2. Part (i) of the task was: Let $y = x^2 - 13x + 42$, what are the roots of $y = 0$?. Below, I compare two approaches:

Automaticity

Solution of quadratic: max two roots.

Factorise, if possible.

As $6 \times 7 = 42$; $6 + 7 = 13$, $y = (x - 6)(x - 7)$

$$y=0 \Rightarrow 0 = (x - 6)(x - 7)$$

So, $x=6$ or $x=7$

Rote habituation

Solution of quadratic: there is a formula⁸⁷ which gives the solutions.

In this case $a=1$; $b=-13$; $c=42$

$$x = \frac{-(-13) \pm \sqrt{(-13)^2 - 4 \times 1 \times 42}}{2 \times 1}$$

Gives, $x=6$ or $x=7$

Not all those solving the problem instrumentally ('rote habituation') need to resort to the formula, guessing the numbers which fit the brackets can also be a rote and not knowledge-like action. I gave the example of the formula use because it is the response I have seen from non-capable students with calculators available to work the arithmetic; it is a reliable method which can be executed without grasp of the entities it involves. "It always works" has been the rationale given by some of these students. The warrantability depends on the form, which in some cases, particularly in the level of mathematics with which I am dealing, can be specified objectively. This is due to the 'local formalism' obtainable in higher school mathematics: i.e., there are parts of mathematics for which consistent formal systems have been developed⁸⁸.

⁸⁷ I use the notations given in the following representation: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

⁸⁸ For example, the propositional calculus and limited parts of elementary algebra and geometry (Wilder, op. cit. p 275, Tarski, 1948). The success of local formal systems is manifest in the technological achievements of the late twentieth century, despite the failure of formal systems to underpin mathematics globally. However, "[there are] parts of mathematics which have been systematically and completely developed" (Wang, 1986, p 325).

7.3.3 *Objections*

The claim that automaticity - an action - can be a sort of mathematical knowledge is surely contentious. Some objections to, or queries about, this claim are answered now.

A. In what sense does the claim ‘automaticity as knowledge’ depend on a notion of a mathematical form being ‘internalised’ by a cognising subject?

Reply An explicit analysis of a mathematical form was given in the case of the Vedic multiplication algorithm. When the sequence of actions of the algorithm is ‘basic’, in Davidson’s and O’Neill’s sense, then I want to say that this form has been ‘internalised’. To explain this ‘internalisation’ further, recall that for such a basic action, intentionality in executing an automaticity is minimal, yet variation of intentionality related to the mental act is possible. ‘Internalisation’ is another way of referring to ‘basic action’ and basic action is related to intentionality.

One of the distinctions between a rote mathematical action and an automaticity is in the characterisation of the corresponding intentionality. Before I explain this, recall that in the case of tying shoe laces, intentionality ‘dies away’ as expertise increases. In mathematical expertise, the corresponding intentionality is more complicated than such a monotonic decrease. A typical mathematical action - like solving equations - is at first laborious, then it becomes rote, then an automaticity⁸⁹. In terms of the intentionality: at first, the form is not internalised and the student’s intention to, say, solve the equation has to be high: ‘what do I do next? Oh yes, put the x s on one side and the number on the other’. Then, these algebraic routines can be performed as rote action. (As Shazdah said: I can do it but I don’t know what it means). At this stage intentionality generally decreases and some behaviourist-type stimulus response has been internalised. When the student solves equations with automaticity there is low intentionality unless something interrupts the flow of the action. (This could be an error or a request for an explanation, for example).

With such an interruption, some justification of the action is required. To do this a mental act, i.e. intention, is required to produce the justification. So the student has to be able to vary from the low intention of routinised action to the high intentionality of justification at the automaticity stage. It is as if, at this stage, there is a flexibility associated with the student's intention which is also internalised.

B. Intentionality seems to be intrinsic to knowledge but lack-of-intentionality to 'automaticity'. How then can it be justifiably claimed that automaticity is (a form of) knowledge?

Reply On the question of intentionality as intrinsic to knowledge, this begs the question about the possibility of the non-propositional knowledge (for which O'Neill argued, 7.2.3). The point is, both from O'Neill's perspective and my own, that we want a form of knowledge to be characterised by minimal intentional action. The point about automaticities is that while the process is invariably carried out with minimum of intention, there is a flexibility in the intentionality for deeper warranting capacity as explained in A..

C. An automaticity seems to be non-propositional knowledge in execution but propositional in justification. How is it possible both to make a distinction and to declare an intimacy between non-propositional and propositional knowledge?

Reply The objection may be answered by bringing in the concept of the potential variability of intention again. The expert Vedic multiplier, say, is able to act instinctively, non-propositionally, on receiving a pair of suitable numbers. But the form itself can also be spelt out in such a way that each step can be couched as a propositional assertion, as was done in 7.3.1.

⁸⁹ Not all students go through each of these stages; some of Krutetskii's capable pupils went straight to automaticity.

D. Supposing it was granted that automaticities can count as knowledge. At what sort of prompt does this 'internalised logical form' manifest itself as observable behaviour (like writing down a sequence of symbolic manipulations as a calculation)?

Reply Firstly, to reiterate: 'automaticity-as-knowledge' is not an observable, as I exemplified with the dividing fraction scenario. It is a mental state, using 'mental' in the sense of Johnson's 'body-in-the-mind': potential actions are part of this state. So, secondly, the lack of precise behavioural criteria which may indicate occurrences of automaticity is not appropriate, even though automaticity is realised in action which may have an observable component.

E. We all make errors sometimes. How can this be if the automaticity - with which a calculation was done - is knowledge?

Reply We all make errors in some individual calculations. This can happen in cases in which we claim to know the procedure involved (my cheque book is full of subtraction errors!). 'Automaticity' signifies a formality in the knowledge - because of the warranting capacity - in which the general structure is more significant than the particular calculation. As Wittgenstein has said about an arithmetic 'mistake': "I must have miscalculated" indicating that the form is more vital than the individual answer. In some sense it is only when we have knowledge of this procedural-form type that we can even recognise errors in execution of such a procedure.

F. The argument that automaticity is knowledge seems to presume the following: if the person acting with automaticity were to be challenged on any aspect of their routinised calculation or process, they should be able to mathematically warrant their action. Is this possible?

Reply Not in practice! The practical impossibility of ever going the whole way to the set theoretic foundation of, say, an arithmetic process does not imply that a warrant was not given. Mathematical forms are apparent at various stages of analysis, (as the Vedic example was intended to illustrate). The question of the appropriate level of formal

scrutiny, in practice, is a question of negotiation for the person with the automaticity and the challenger (who claims the former is acting instrumentally).

7.4 CAS, calculus and automaticity

Keeping your eye clearly on the mathematics requires an extremely fluent use of the technological device (observed of Douglas Butler⁹⁰). An extremely fluent use of the technological device does not imply that you have your eye clearly on the mathematics (observed of some PGCE students⁹¹).

As an application of these ideas on automaticity, I shall discuss two propositions relevant to higher school mathematics:

- a) *On mathematical performance and perception of structure*: A student of calculus, if s/he has a CAS available, does not need to know the structural properties of a one variable differential operator, like the 'product rule', to be able to solve several classes problems. This may effect the range of problems capable of being solved.
- b) *On CAS and 'capability'*: For not-capable⁹² students, the use of a personal technology CAS may militates against their achieving automaticity of 'symbolic manipulation' (typical in mathematics A level). For capable students, a personal CAS could well enhance the possibility of their achieving this automaticity.

⁹⁰ Douglas Butler, a teacher at Oundle School, gave an impressive demonstration of mathematics teaching with IT at The Mathematical Association conference, University of Strathclyde, 1997.

⁹¹ In particular the non-mathematics graduates on the 2-year mathematics secondary PGCE to whom I taught calculus.

⁹² I do not wish to stereotype students' ability, (in the sense of Ruthven 1987), but use Krutetskii's term to denote practising teachers' recognition of differences in capability.

Chapter 7: Mathematical Action

Before discussing these propositions, some brief notes on CAS are in order. Firstly, like electronic calculators, they are not primarily pedagogic devices⁹³. Developed for commercial or academic interests, (Oldknow and Flower, 1996, pp 52-4), the manufacturers clearly had, and continue to have, economic interests in widening their market into the educational domain. The glossy advertisements in our professional magazines are there first and foremost to increase the profits of these transnational companies. Although the initial motivation for developing CAS was not pedagogic, it did not take long before the enterprising manufacturers caught on to the huge educational market, so adaptations took place and substantial investment has now taken place to ensure that machines are developed specifically for students (in school, F.E. and H.E.). The TI 92 is a particular case of a relatively inexpensive machine which can perform a range of different software packages relevant for an advanced mathematics student.

The extent to which mathematics teachers and educationalists were born to accept these technological innovations, sought these helpful devices, or had these commercial intrusions thrust upon them, is a socio-political story outside the scope of this thesis. My point here is only that there is no a priori reason for such, economics driven, technology to benefit mathematics learning. And this is not suggest that I am trying to imply that their use need be harmful in mathematics learning either!

The second preliminary point, is that it is relatively difficult to become adept at using these advanced calculators and CAS packages. Oldknow and Flower point out "Most CAS work in rather idiosyncratic ways, which can appear confusing, particularly for students with weak algebraic skills" (op. cit. p 3). The time investment required to get to the position where this technology can be used productively can be considerable. This has been recognised in some recent course materials, for example, the recent Nuffield Advanced Mathematics took up about half of its Book 1 on specific teaching on how to

⁹³See Pimm, 1995, pp 76 -87. Pimm points out that these technological calculating devices do not have the 'transparency' of mechanical calculating devices like the *soroban*, (p 80).

use an advanced (many function and graphic) calculator. In classroom, much troubleshooting is often required to help students enter and read expressions in Derive.

Some students are intrigued by their calculator and explore its functions as an interest. Most students merely contend that facility with their calculator is essential for good exam performance. The calculator is another thing to learn about in maths lessons and it can help get the answer to some calculations easily. Is the push for technological proficiency in the same direction as the push for mathematical knowledge? A priori there seems no reason to suppose that an increase in CAS or calculator technological proficiency increases mathematical knowledge⁹⁴. That is not the same as pretending that the unbounded data obtainable from the calculator or CAS does not offer the learner a wealth of experience, just by virtue of being information. This 'rich environment' may be fertile for learning⁹⁵. But 'proficiency' does not imply that the putative learner, in ecological realist terms, "perceives affordances" (Kitcher, 1984, p12) in the CAS or calculator's output. There are still two conceptually distinct, yet intersecting domains, IT and mathematics. It does not follow that learning IT with or through mathematical applications is mathematics learning. My point here is that IT learning involves investment. IT can provide splendid images, information and experiences from which students may find it easier to generalise - to act mathematically - but, there again, the student may not 'see through the screen' (Mason, 1993b), and, like any other aid, the student is locked in the particularity of the device.

7.4.1 'The product rule': performance, and perception of structure

The 'product rule' of differential calculus is a standard topic in higher school mathematics. In section 7.4.1.1 I consider the role of this rule in 'problem solving' and in

⁹⁴ It has been argued, (Laborde 1993, p40) that computer systems can change the mathematical objects under study. Laborde's examples came from cabri-géomètre, but there may be analogies for the algebraic objects which are CASS' focus.

⁹⁵ Ruthven (1990) reports on the use of graphic calculators having a significant positive effect on students' capacity to associate an appropriate algebraic form with a given graph of a function.

further learning when a CAS is available. In 7.4.1.2 the role of CAS in learning about this rule is discussed.

7.4.1.1 Problem solving and perception of structure.

While “not all mathematics is problem solving” (Wang, op. cit. p227) appreciation, routine execution etc. count as well as application, much of the vaunted purpose of learning mathematics is to be able to use mathematical techniques in solving problems. As Wittgenstein said, “the *use* of these symbols is a criterion of their meaning” (1976, op. cit., p81). This purpose-focus has always been recognised, but had particular expression in the early 1980s through publications like the Cockcroft Report and curriculum development materials like those from the Spode Group, for example. If problem solving is the prime purpose of mathematical activity, of what relevance is a student's knowledge of the product rule if her CAS can return an answer to any differentiation involved in her model of a problem? To discuss the desirability of the student being familiar with the product rule, I look at detail in solving a particular problem. This is a closed problem, a typical A level example, (from SMP 16-19). It has a definite answer but does have different possible approaches to finding that answer:

A mathematical ornament consists of a cone inside a sphere of radius 5 cm. such that the top and the perimeter of the base of the cone touch the sphere.

Design the ornament so that the cone has maximum volume.

Letting r be the radius of the given sphere, a reasonable move is to set the radius of the cone as x . This gives the 'model' for the volume of the cone as

$V(x) = \frac{\pi x^2}{3} \left(r + \sqrt{r^2 - x^2} \right)$. If you know that turning points can be found by setting the

derivative to zero, then, if you are familiar with a CAS (like Derive), the value of the required radius can be found without any further mental calculation. You do not need to be aware even that differentiating products needs a little care.

Alternatively, the volume expression does not have to be set up quite like that. By using a different variable, y , the distance of the centre of the circle to the centre of the cone's base, the volume function becomes $V(y) = \frac{\pi}{3}(r^2 - y^2)(r + y)$. This is easier to differentiate. And this can be seen directly by one with suitable algebraic automaticities: possibilities in setting up the model were related to easier or more difficult actions in calculation⁹⁶.

That is, understanding is not only 'understanding for or within application' but also understanding of structure. Understanding of structure, such as that of the one variable differential operator on a product, includes the perception and action properties of automaticity. Without these capabilities, as is generally recognised in the psychology of mathematics education, the student cannot develop to the next stage of abstraction. As another example, consider the following: if $\hat{r} = \cos \theta \hat{i} + \sin \theta \hat{j}$ is a unit vector in polar co-ordinates with θ a function of time, then its length (and length-squared) is 1. This can be expressed as: $\hat{r} \cdot \hat{r} = 1$, which can then be differentiated: $(\hat{r})' \cdot \hat{r} + \hat{r} \cdot (\hat{r})' = 0$. To be able to 'see' the expression $\hat{r} \cdot \hat{r} = 1$ as a product with formally implicit differential $(\hat{r})' \cdot \hat{r} + \hat{r} \cdot (\hat{r})' = 0$, is to have an automaticity with regard to the operator's structure. To be able to 'read' the result as implying that the original unit vector (position) is orthogonal to its derivative (velocity) requires a similar facility with the scalar product.

Another result obtained with these notations, together with the product rule, is the expressions for the radial and transverse components of acceleration. A way of getting these formulas is to differentiate twice (with respect to time) the expression for a general position vector in polar co-ordinates: $\underline{r} = r\hat{r}$, where r and θ are both dependent on the same parameter t . In my teaching experience, I have found that the differential calculations for these components of acceleration are practically meaningless to students who have not 'internalised the rhythm of the product rule'. The product rule is being

⁹⁶The text book suggests a trigonometric variable as well, so that the student can compare methods.

used here structurally and to understand the calculation (let alone produce them) the student requires more than recognising a potential application of the product rule to $\underline{r} = r\hat{r}$. In other words, automaticity involves structural understanding as well as computational competency. Without such automaticity the possibility of creative applications is limited. At a different level, the question of how arithmetic functioning is intrinsic to mathematical progress is also a matter for current debate, (see SCAA 1997).

7.4.1.2 *Learning about structure with CAS.*

There are many ways whereby a 'rich environment' can be designed by the teacher, using CAS, to facilitate the students' generalising and finding a new mathematical concept or property. Examples can be found within the "activities for the classroom" in Oldknow and Flower (op. cit. pp 13- 43) or in the articles in MicroMath (Autumn 1993 pp 17 - 42).

For example, an outline of a typical CAS lesson on 'differentiating products' is to be found in Oldknow and Flower (op. cit. pp 28 - 29). The exercise that is given is to be able to spot the occurrences of the derivatives f' , g' and of the original functions, f , g , of the product fg , an example of which is $x^3 \sin x$, in the CAS returned derived function $(fg)'$. What the CAS does is to generate data for the student and, like the 'think of a function' game used throughout the school years, the student is to conjecture the rule that is the source of the generated data. The CAS will be able to give impartial and non-judgmental feed back on the quality of the students' conjectures, which many students find helpful because it is non-threatening. The resulting target knowledge is, at least, recognition of the formula: $(fg)' = f'g + fg'$. The investigation employs the receptive skills of recognising patterns of notations, but does not require any productive generation of derivatives.

In their commentary on their product rule CAS activity, Oldknow and Flower note:

⁹⁷ This 'process-object' duality has been the topic of a substantial body of research in mathematics education. Principal contributors include Tall (1991) and Sfard (1991, 1994), for example.

"It is frequently argued that computer algebra systems make redundant the mastery of many of the algorithms in calculus (e.g. product rule, quotient rule, integration by parts). Therefore, it may seem an anomaly to use CAS to discover a rule which it is no longer essential to know if you have access to CAS." (ibid. p 29).

I suggest that these rules provide a structural understanding of the basic differential operator that is needed - as an object (see chapter 6) and an automaticity- as a foundation for more advanced mathematics. In the case in question, the structure of the product's derived function is $(fg)' = f'g + fg'$ and the process involves 'seeing' the derived function of a product in this form - and being able to 'do it'!

I have reservations about Oldknow and Flower's assertion that:

"the consequence of the availability of automated symbol manipulation will be a shift away from manipulative facility and towards the ability to construct and interpret symbolic expressions." (ibid. p 47).

I do not agree that 'the ability to construct and interpret symbolic expressions' can be developed without 'manipulative facility'. I'm sure that they would argue that the phrase 'a shift away from manipulative facility' does not imply 'no manipulative facility' ! Nevertheless, my point is that without some manipulative facility, the structure has little chance of becoming embodied as an automaticity, which as I argued in 7.4.1.1, is central to structural understanding.

7.4.2 CAS and capability

To make a case for proposition (b), I first want to recall Krutetskii's distinction between the capable and the not-capable in this regard:

"Able pupils are distinguished by a rather pronounced tendency for the rapid and radical curtailment of reasoning and of the corresponding system of mathematical operations....it even begins to appear in the first problem of a new type to them." (Krutetskii, 1976, p265).

The example of the binomial, given above, $(27x^3+27x^2+9x+1)$ is seen as $(3x+1)^3$ whereas $8y^3+8y^2+4y+1$ is seen as not having a representation as the cube of $(2y+1)$, is a good one to have in mind here. It is this immediate - and not necessarily aware - grasp of the structure that is a key point of my argument. A student who instinctively works structurally (which is part of Krutetskii's characterisation of 'capable') will already have the germ of the embodiment of, say, binomial expansions. If such a student has a personal CAS, the data that s/he will be able to receive from the CAS can build on this embodied germ itself. The personal aspect of the technology facilitates easy checking and exploring of related expansions.

On the other hand, a student who has no basic structural understanding and has a personal CAS will receive the data from the machine as particulars. Although s/he may well construe a pattern in these data, that pattern will become linked with his/her use of the machine as data generator. So what becomes embodied is the facility with the machine and what is not called to account is the structural understanding of the essence that the capable student got free at the outset. Because "in the usual method of generalising, the average pupil perceives the generality of features by contrast, the [capable] pupil infers the features' generality from their essentiality" (ibid. p 259), so the capable pupil will be able to use the CAS data to build detail on essential structure - hence developing this automaticity- , the less capable student will be able to generate aspects of the perceived contrast through use of the machine - hence developing their machine use automaticity.

Krutetskii's research was done in the 1950s and 60s, well before the development of readily available calculators, let alone CAS. He quotes other researchers on the issue of curtailment of reasoning; they assert that it comes about only by practice of exercises. (ibid. pp 264-5) Krutetskii concurs with their view when referring to the non-capable student, although he disagrees when it come to the capable ones.

The issue raised by this is the distinction between practice and data in algebraic manipulations given the availability of hand held CAS. I suggest that an experimental, data based, CAS approach to teaching average students algebra could well reinforce the

view that Krutetskii found with non-capable students that “many were convinced for a long time that algebra was an operation with letters, which can be added, multiplied, or raised to a power using definite rules” (ibid. p 254). This is because, as quoted above, the average student does not perceive essentiality but is able to distinguish features. The use of a personal CAS will only reinforce the feature spotting rather than the structure. Furthermore if the sort of exercise is advocated where students explore data - say derived functions - as discussed above, they will get limited opportunity to do the workings themselves as a mental (rather than perceptual) process. The consequence of this is that ‘basic actions’ become operations on the machine, rather than mathematical automaticities.

7.5 Summary: only in mathematics

Fluent mathematical activity is a phenomenon familiar to mathematical practitioners - because they experience it - and to mathematics teachers - because they are involved with developing this facility in others. But how is the capability to calculate or to do algebra etc., related to mathematical knowledge? In this chapter I have tried to give an account of ‘automaticity’ and claim it is a form of mathematical knowledge. At first sight, to execute an automaticity, appears to be to exhibit a non-propositional basic action. Such need not be knowledge. People can be trained to instrumental mathematical activity, to execute ‘rules without reason’, and this does not satisfy the warranting criterion required if a knowledge claim is to be made. So the concept of ‘automaticity’ must have a warranting dimension if any knowledge claim can be made. This can be done without introducing a new item of knowledge, because, in the unique knowledge domain which is mathematics, the warrant is both the structure of the action and the justification.

8. Chapter 8: Conclusion

The untamedness of numbers is in their order, resolving upwards into a speculated beauty. Too close and language fails. (Jeanette Winterson, 1997)

8.1 Overview

Every discipline has its particular ways of reasoning, objects of study and instinctive-like routines. Within these pages has been an investigation into the nature of mathematical warrants, objects and ‘automaticities’ as they pertain to higher school mathematics. This investigation has entwined a philosophical analysis of mathematical warrants, objects and actions with an application of these concepts with regard to learners of the subject. On the general level I have made a case for:

I. the distinctiveness of mathematical warrants

II. the existence of mathematical objects

III. the knowledge-status of mathematical action

Applying these ideas to mathematics in education I argued that:

- i. ways of reasoning at this mathematical level include deduction, quasi-empiricism and visualisation, and that students need not only to learn these processes, but also that these processes are the ones which serve to justify mathematical propositions
- ii. ontological commitment to the content of higher school mathematics is integral to a student’s progress and a consequence of realism in mathematics
- iii. learning mathematics involves developing a capability to execute some mathematical procedures with ‘automaticity’

While the discipline under scrutiny in this thesis was that of higher school mathematics, the method of scrutiny was through application of philosophical reasoning. This is an unusual approach in mathematics education. The only other writer of whom I am aware

Chapter 8: Conclusion

who explicitly uses philosophy reasoning applied to mathematics education is Paul Ernest (e.g. 1991, 1997). Ernest is well known as a champion of a ‘social constructivist’ philosophy of mathematics; that ‘objectivity is social’ is fundamental to his theory. Except for a brief rebuttal in chapter 6, I have not constructed this thesis as an argument against Ernest’s position but tried to develop an alternative ‘objectivity is Gaia⁹⁸’ theory. There are two main reasons for this. Firstly, head on, ‘adversary’ battles, as I reported from Moulton’s work in chapter 2, are unlikely to produce fresh thinking or to convince anyone in the heart. The best adversarial reasoning can do is to ‘score points’. Secondly, it was important to communicate the work of realist philosophers who, barring Kitcher and Lakatos, Ernest had not mentioned in his 1991 book. The application of the work of philosophers such as Goldman⁹⁹, Maddy, Resnik, Zheng and Giaquinto and others to issues of mathematics in education is intended to be part of the contribution of this thesis.

The main body of this chapter begins, in 8.2, with a review of the comments made after doing the mathematics presented in chapter 3. Then I revisit the themes from the ‘teachers dialogues’ of the introduction, in 8.3. In 8.4 the theoretical ideas of the thesis are applied to some A level mathematics questions. Section 8.5 is a final summary.

8.2 *On the comments arising from doing mathematics*

In sections 3.2.n.4 in chapter 3, I listed some seventeen comments in all which were ‘for further philosophical investigation’. Many of the ideas in these comments were used in expressing the three-part thesis which was expounded and defended in chapters 5, 6 and 7. The most important themes from those comments were on:

- ◇ the way by which mathematics may be known. (From 3.2.1.4: the role of perception, paradigm examples; from 3.2.3.4: role of technological enhancers, what determines

⁹⁸ Gaia is the Greek goddess of the earth. The ‘Gaia hypothesis’, (termed thus by Lovelock and Margulis), is the idea that the whole (physical) world is interrelated, from rocks to biological organisms (including human beings). (Kapra, 1982, p 307)

⁹⁹ Dylan Wiliam has used ideas from Goldman (1976) in Wiliam 1994.

Chapter 8: Conclusion

truth of a mathematical proposition; from 3.2.4.4: new ways of grasping the same phenomenon can occur.)

- ◊ the relationship between what is experienced within the world - particularly the material world - and mathematical knowledge. (From 3.2.2.4: the notion of a mathematical object; from 3.2.3.4: the relationship between lived experience and mathematical experience; from 3.2.3.4: the tension between rationalism and empiricism.)
- ◊ how action, in doing mathematics, is linked to mathematical knowledge. (From 3.2.1.4: the relation between action and understanding, and between algorithms and processes; from 3.2.2.4: the power of notation.)

There are inevitably some loose ends from those comments, for example the function of aesthetics, the notion of a change of mental state and a thorough investigation of the role of notation. However, those comments contributed to provoking the theses on the distinctiveness of mathematical warrants, on realism in mathematics, and mathematical fluency as a form of knowledge, respectively.

8.3 *Back to the themes of the 'teachers' dialogues'*

One of the purposes of a philosophical piece of work is to try to unravel puzzles arising from experience. In the first chapter, the teacher-dialogues illustrated various puzzles arising from higher school mathematics under the themes of belief, mathematical objects, notation, proof and *wis en zeker*. As I said in the introduction, these themes are woven through the work as a whole. Nevertheless, the themes of belief and proof are predominant in chapter 5 and chapter 6 is specifically on 'mathematical objects' which includes the *wis en zeker* theme. The theme of the role of notation, together with belief, proof and the objects of mathematics is to be found in chapter 7.

Looking back to the themes more specifically: From chapter 5, we find that to get to know any propositional mathematics a belief about such a proposition must be formed. Generally, warrants justify the truth of propositions of the sort abounding in higher school

Chapter 8: Conclusion

mathematics. For a student to have a warranted belief in a mathematical proposition, specifically mathematical warrants are required. These include proofs but, any teacher knows that proofs do not (often!) have the power to convince. A classroom challenge is to get the learner both to be convinced by a mathematical proof and to grasp that producing mathematical proofs is paradigmatic mathematical argument.

From chapter 6, we find that human perception and physical activity form the foundation from which relations, patterns and abstractions - which are raw mathematics - are then classified and codified. With this foundation, mathematics need not be conceptualised as *completely* divorced from the physical world. A physicalist-realist mathematical ontology resolves the paradox of 'real abstractions': properties and relations - which yield mathematical objects - are, initially, properties of and relations between physical entities. Relations between the relations provide more possibilities for mathematical abstractions. These relations can include the relationship of negation thus permitting possible worlds which are independent of physical instantiation. Consequences of the possible worlds are bound by mathematical reasoning rather than by perceptually based experience.

From chapter 7, we find that mathematical knowledge is not just of propositional form: the notations and symbols of mathematics can serve as a 'blind-man's stick' within fluent mathematical activity, (typically algebraic manipulation and calculation).

The response here to the themes of the introduction is brief because detail of the explanations is in the body of the thesis as a whole.

8.4 A sample of higher school mathematics analysed in terms of warrants, objects and actions

Those studying higher school mathematics within school or college are usually assessed on their competence with the content. Indeed for many students the purpose of studying the subject is to obtain some kind of certification which allows them to progress towards a personal or career goal. The evidence for whether Joe knows this mathematics he has been studying is assessed by his teachers, possibly his own sense of whether he has

understood, but these are both influenced and overshadowed by the public exams which certify his knowledge. Public exams have been subject to change in recent years¹. The rhetoric that the aim is understanding, rather than 'just' doing, has crept into the system. The typical student, Joe, will probably consider that he has understood the mathematics of his course when he can do problems of the type on which he will be tested.

The relationship between assessment and knowledge is immensely complicated. A thorough analysis would have to delve into psychology, sociology, politics as well as the history of education. Clearly, this task is outside the present work. But a limited analysis of what the knowledge is which is intended to be assessed can be done using the philosophical ideas of warrants, mathematical objects and automaticities. This may be achieved by taking assessment items, at the mathematical level in question, and specifically locating objects and warrants intrinsic to this assessment. Two A level questions and a pedagogical discussion starting point are analysed in this way. The A level questions are typical ones from algebra and calculus, single and further maths respectively. The problem situation is one which often provoke students to question their 'common sense' understanding.

8.4.1 A level algebra question

A level question 1, taken from AEB November 1989, Common Mathematics Paper I:

¹ The idea of testing 'understanding' is an acknowledged aim both in the course-work assessments of, for example, GCSE and some GCE A level exams, and also in some of the English and Welsh National Curriculum SAT tests.

A geometric series has first term 4 and common ratio r , where $0 < r < 1$. Given that the first, second and fourth terms of this geometric series form three successive terms of an arithmetic series, show that

$$* r^3 - 2r + 1 = 0$$

Find the value of r .

What mathematical objects are dealt with here? What are suitable warrants? Where might automaticities 'click in'?

Objects. Within any given mathematical 'situation', as the example just given, there can be construed a myriad of constituent concepts which, in terms of the discussion of chapter 6, could consistently be classified as 'mathematical objects'. In the particular 'situation' reproduced, I suggest that part of the examiners' intention is to assess students on the two types of regular progression called (here) 'geometric series' (GP) and 'arithmetic series' (AP). As mathematical entities, these progressions have property-defining structure: a sequence of numbers of either of these types can be specified by just two real numbers and an operation. (These parameters are the first term¹⁰¹, the factor of 'increase' and the operation of 'increase'.)

An ontological commitment to GPs and APs is not necessary to do this question - formulae for the sum and general term of these two sorts of sequences are in A level formula books. A student may well be able to act as follows: write down the expressions for the second terms by plugging in '2' for n in the given formulas and equate them, then, write down and equate the third term of the AP with the fourth term of the GP. That a student may be able to achieve satisfaction on the assessment item without a grasp of the

¹⁰¹ Any term may be used to specify the sequence, but then an extra natural number parameter, n , is needed to say which term this is.

Chapter 8: Conclusion

entities involved says more about the assessment item's design than the intention in setting it or the mathematical objects it involves.

Warrants. This problem involves two basic propositions: (i) that r satisfies the given cubic; (ii) that $r = \frac{-1 + \sqrt{5}}{2}$. What are mathematical warrants for belief in these propositions?

For (i), a standard warranting of the proposition, requires that an unknown - the 'common difference' of the AP - is eliminated from the two simultaneous equations. Typically, this is done by the equating of the second terms and equating of the fourth and third terms of the GP and AP respectively. Deductive reasoning is used to give an expression for r from the two given equations in two unknowns. Despite this formal mathematical warranting, an authoritative warrant for belief is employed when the student, having carried out the deduction, checks back in the question and finds to his relief his answer is confirmed.

For (ii), the problem solver requires some notion of root including the possibilities for how many roots a cubic can have. The root required in this case is a real number which, when it is cubed, is equal to two times itself minus one. A student can get thus far without a method for finding a value for r . But if he finds an r (which satisfies this relation) his method of finding is bound in with the warrant by which it is believed that the particular r found is indeed a root. Mathematical knowledge - in this case knowing the proposition (ii) is true - is subject to mathematical warranting.

Here are some warrants which a student might use to show the truth of (ii): (a) he may guess; (b) experiment graphically; (c) use the formula for solving cubics of $x^3 = px + q$ type; or (d) extract one root and solve the remaining quadratic. Each of these methods has its own warranting power: For (a), a guess is only as good as the check; for (b), a graphical experiment is akin to a perceptual warrant; for (c), a mechanical substitution into a received formula is only as convincing for the student as his ontological commitment to the formula; for (d), extracting the whole number root and then

factorising the remaining quadratic is a three stage method: the whole number root is located, the linear factor is extracted, then the remaining quadratic is solved.

Automaticity. Algebraic manipulation, whether as an automaticity or as a laboured reflective process is required to prove the relation satisfied by r , $*$, and that this equation is satisfied by the specific value r . There is no guarantee, nor requirement, that mathematical action-knowledge of the automaticity-type needs to be used in solving this problem. However, if every manipulation or deduction were executed without automaticity, it is likely that the student would not finish the exam¹⁰². In other words, the time constraints of the assessment would be the only lever to indicate that some automaticities were part of the students mathematical knowledge even though it would not be apparent which ones.

8.4.2 A level calculus question

A level question 2, taken from NEAB 1998 Specimen Questions, Paper 6.

A curve in the x - y plane is such that for $x > 0$ the tangent at every point (x, y) on it intersects the y -axis at $(0, x)$. Show that

$$x \frac{dy}{dx} = y - x.$$

Hence find the equation of the curve which possess this property and which passes through the point $(1, -2)$.

What mathematical objects are involved in this problem? What are suitable warrants for belief in the propositions which are required to be proved? Where might automaticities contribute to knowledgeably solving the problem?

¹⁰² In the discussion paper, (SCAA, 1997), time constraints are suggested as a means by which effective mental strategies - potentially automaticities - are encouraged (p18)

Chapter 8: Conclusion

Objects. A particular curve in the x - y plane need not be conceptualised as a mathematical object for it has, potentially, a perceptual existence and so can be construed as merely a perceptual item. However, a curve defined by properties owes its existence to relationships between those properties. A student aiming to solve this problem will find it difficult without an ontological commitment to property-defined geometric objects. In this question the property-defined geometric objects are the curve and the tangents to the curve. The curve only exists because of a property of its tangent. And the tangent only exists if the curve does! In this case, the properties of the curve are (a) a particular relationship between the rate of change of the y value with respect to the x value with those x and y values; (b) a particular point it passes through.

Warrants. Like the question above, the student is asked here to prove two propositions:

(i) that $x \frac{dy}{dx} = y - x$; and (ii) that $y = -x(\ln x + 2)$, given $(1, -2)$ lies on the curve. The

first proposition is no more than an unravelling of the meaning of the question and expressing it in a particular notation. The warrant for belief is basically authoritative - *given* a grasp of tangent to a curve - 'have I got the correct symbols?' The second proposition is the main content of this question. In a similar fashion to the first question, the method used to find the equation of the curve is bound in with the warrant by which it is believed that the particular $y = y(x)$ found - (this curve does happen to be a graph of a function) - is indeed the correct one.

Consider analogous methods to the ones hypothesised in the other question: (a) guess-check; (b) graphical experiment; (c) formal use of integrating factor; (d) experimenting with symbolic forms to get an exact ODE, then integrating both sides of the equation with respect to x .

For (a), while a guess (or approximate guess) of the function seems unlikely, the great thing with differential equations like this is that if the solution fits then the problem is solved; the explicit check is crucial to this warrant, without a demonstration that the properties are satisfied there is no reason to believe an hypothesised function has the properties required. It is a similar situation when a graphing experiment or numerical

methods are used in method (b): the relational properties need to be confirmed no matter what tools are used to help the student to come up with an hypothesised solution. In the case (c), of the formal symbolic method, this could be performed by a suitable CAS, but the student will probably still have to recognise the form the CAS can accept the differential equation and organise the input accordingly. Nevertheless, the warrant would be a perceptual matching of notational form together with the authoritative warrant due to the machine's capabilities. The formal symbolic method can be employed 'by hand' of course, in such a case the warrant for belief in the solution's veracity is comparable to the warrant in the case (d) where an integrating factor is found though some experimentation with the symbolism. However, the crucial conceptual move here for the student is to recognise the importance and procedural function of manipulating the equation into an exact form from which a symbolic integration can be worked; deductions of this type (integrating to get a solution) require premises (the given differential equations), and the structure of the given equations legitimates the very deduction of the solution function.

Automaticities. Again, there is no point of the exercise which requires an explicit automaticity, yet without some such fluent functioning it is difficult to imagine that the task could be completed. (In the case that a CAS does the difficult bits, the point is that the machine operates automatically - what else can a machine do!) The translation of the problem into a symbolic form, to get proposition (i), is perhaps one part of the process of solving the problem where it is difficult to be reflective about every aspect of the reformulation and still finish in a finite time. It would not be unreasonable to expect a student tackling this question to know automatically, 'in his body', that an equation of a straight line has form $y = mx + c$. Furthermore, for a given point, (x_0, y_0) , the question-information says that $c = x_0$, so a procedural automaticity may be realised in making the substitution $y = m_0x + x_0$, where m_0 is the gradient function evaluated at the arbitrary given point.

8.4.3 Problem situation from Mechanics

A super ball bounces to half the height from which it drops.

We cannot see microscopic bounces - how do we know if the ball ever stops bouncing?

This problem does not consist in a request to prove specified propositions in the way the two A level questions did; it can be used pedagogically to start a discussion from which such a proposition may be conjectured. A discussion may include choosing the mechanical model, specifying the initial conditions and agreeing an appropriate symbolic representation. The outcomes of an investigation are less definite than of a closed question, but the question of what the objects, warrants and automaticities might be can still be posed.

Objects. The ball is not a mathematical object but, in some sense, the basic kinematics equations for constant acceleration are! (An assumption has been made that the choice of the model is $\ddot{r} = -g\hat{j}$). The kinematics equations are objectified in the sense that the relations between a constant rate of rate of change, the dependent and independent variables and the rate of change, are deducible from the notion of rate of change. The other significant mathematical entity to contend with in this situation is countable infinity and its application in an infinite series.

Warrants. A proposition which may be conjectured from this starting point is (i) $\sum_{j=1}^{\infty} t_j < \infty$, where t_j is the time taken for the j^{th} bounce, i.e. the ball stops bouncing within a finite time. The mathematical statement, (i), is that an infinite number of nominally physical items (times, t_j ,) may be combined to make a finite one. A warrant, again, is essentially the method used to show that (i). The proposition may have been conjectured because of the perception that the ball does physically seem to stop bouncing. In terms of the vocabulary used in the A level examples, this method of getting the result 'it stops bouncing' was by using a perception warrant.

Chapter 8: Conclusion

The more specific proposition, (ii), is $\sum_{j=1}^{\infty} t_j = \left(\sqrt{\frac{2s_0}{g}} \right) \left(\frac{1+\sqrt{2}}{-1+\sqrt{2}} \right)$, where s_0 is the initial height. The infinite sum is the sum of a GP, the terms of which are obtained from the equations of motions of the basic mathematical model employed. Algebraic reasoning and manipulation is required at this stage. The discussion reverts to that of the first A level question.

Automaticities. There are two distinct aspects of problem solving of this type: setting up the model and working through the calculations consequential to that model. Possibilities for automaticity in the calculations part of the problem solving have been already discussed. The aspect of applying a model is really a scientific enterprise rather than a mathematical one.

8.4.4 Summary

This analysis of some higher school mathematics problems illustrates how a philosophical theory can be used in a practical context. The philosophy reasoning which I have used and the subsequent theoretical ideas I have presented, may not be everyone's cup of tea, but the process of thinking through and expressing these ideas has clarified, for me, the nature and (mathematical) purpose of higher school mathematics teaching.

8.5 Directions for further work

The results of the research presented within this thesis are predominantly conceptual ideas about the nature of mathematical knowledge at a beginning level of abstraction. The research base was literature from English language-medium philosophers (although some translated sources were referred to). As the project was, in part, to apply philosophy to mathematics in education, there are bound to be many more avenues to take than I have been able to do justice to within this particular thesis. The following three suggestions are intended to give a taste of possible future philosophical research in mathematics education compatible with the themes within this work:

Chapter 8: Conclusion

Linguistic realism as a philosophy of mathematics applied to 'coming to know' in mathematics. Azzouni's (1994) novel perspective on the nature of mathematical knowledge merits application in education. He argues that mathematics is neither realist in the 'physicalist' sense nor is it 'anti-realist' in the conceptualist sense; mathematics is ontologically unique. Azzouni's commitment to offering an ontology compatible with mathematical practice bodes well for his theory's being applicable to mathematics in education.

The North American pragmatic tradition and the influence of nature on cultural abstractions. The 'pragmatic' philosophical tradition is associated with American writers from Dewey to Putnam; at its heart is the question 'how do we function?'. Different mathematical functionings develop from the same physical world but from peoples with different social concerns. The Navajo conception of space, (e.g., Ascher, 1991), has a quite different formal expression from the dominant quasi-Euclidean conception. Such geometric 'different expressions of reality' merit explanation in terms of peoples' pragmatic interaction with the 'world'; what does 'objective' mean, what would be 'objective mathematics'?

Realism and anti-representationalism, is this possible? Donald Davidson's fundamental question to himself is 'how does thought begin?' (Davidson, 1997S). His work cuts across linguistic analysis, epistemology and ontology, and is labelled 'realist' by some (e.g. Dummett, 1992) and the contrary by others (e.g. Evnine, 1991). Rorty (1990) classifies Davidson as 'anti-representationalist' because Davidson eschews the notion of epistemic intermediary, like perception, being a cause of knowledge. Because the central question I worked with was, paraphrased, 'how does mathematical thought begin?', a more thorough application of Davidson's theories, than those touched upon in chapter 7, could be fruitful. Specifically, Davidson's conception of the essential triangulation required for knowledge - putative knower, another putative knower, world - seems appropriate to try to apply to issues specifically concerned with mathematics in education.

The overriding theme in these projects is the focus on the fuzzy and fiercely complicated boundary between the given 'external world' and human artefactual knowledge -

Chapter 8: Conclusion

specifically the abstract discipline of mathematics - which I consider fascinating and problematic. The conceptualist answer diffuses questions about this boundary by asserting that 'there is no 'given' external world; the world is what 'we' make of it', i.e., there is no such boundary to try to understand. How 'we' do 'make knowledge' is the basic conceptualist epistemological problem, but they have solved the ontological question by fiat.

Another possible set of future research projects could be developed from deepening tracks already pursued within the thesis. I shall briefly give two examples to help bring these sorts of projects into focus:

Mathematical practice: how can Kitcher's conception be used within pedagogy? Kitcher's five-component conception of 'mathematical practice', together with the 'rational transitions' integral to the practice, may be able to be applied to mathematics in education. Two ways which this potential application might be investigated: firstly, the question of the compatibility of his 'rational transitions' with 'cognitive development' needs to be addressed; secondly, curricula expectations in student progression (what mathematics proceeds from what in the syllabuses) may be assessed against the 'rational transitions' of Kitcher's theory.

Automaticities: mind or machine? Educational issues currently make news in Britain. Of the most newsworthy of educational issues is the state of British children's mathematics. The question of what children can 'really do' is paramount. What an individual child can really do must surely be a function of that child's individual brain? Hence the up-surge in interest in mental arithmetic: Suzie's mental calculation is truly hers because 'it happened' in her brain. But what does 'happen'? A rote response or fluent action? When does executing a particular mental process constitute knowing? And when does subordinating calculation to a mechanical device suggest facility rather than lack of facility? Such questions can be addressed using both philosophy reasoning generally and the theoretical ideas of chapter 7. When to use machines to enhance or replace student automaticity may, using such analysis, have a more rational response than presently heard in the so-called debate betwixt the 'technological enthusiasts' and the 'traditionalists'.

8.6 *"Image and reality: the oldest distinction of all."*¹⁰³

Some years ago a student of mine, Jennifer Wilson, did a GCSE project on permutations and combinations in bell ringing. Jennifer was always keener on music than mathematics and the hope behind her pursuing such a project was that her interest in bell ringing would give her insight into elementary group theory. Some years later a question was raised in mathematics education circles, (by Dick Tahta, I believe), which asked whether there could have been mathematical bell ringers before permutation theory was developed. Could Jennifer have been mathematically au fait with permutations instinctively because of her bell ringing interests? -well that was my hope as her teacher! And this is a practical example of a teacher's application of the philosophical mathematical realism for which I have been arguing. This is because, when it comes to bell ringing, there may well have been 'mathematical' bell ringers pulling harmoniously with 'non-mathematical' bell ringers¹⁰⁴. This proposition, of course, cannot be checked. But I do not want to deny the possibility of pre-cultural spontaneous 'mathematical' activity. (However, one cannot expect to find evidence of such activity because the communication channels are not there, for the pre-cultural hypothesis implies that there would be no extant evidence of permutation theory). However, those bell ringers may have had thought patterns which, if they could have been communicated, would be recognisable as 'mathematical'. And these 'thought patterns' came about because of their mental and physical interaction with a specific environment. The musical image veils a mathematical reality.

There needs, finally, to be some comment on the relationship of academic writing, such as this, and the wider sphere of moral values, for these inevitably colour the intention of the writing. What I have presented here is an account of the nature of mathematics

¹⁰³ Pat Kane (1998). Kane was writing about media image and political manipulation. He concluded his article by saying: "But the world was never as simple as image versus reality." His sentiment of reality being desirable but illusive, merged with language, culture and human relationships has an analogy to my story about mathematics.

¹⁰⁴ The question of whether, or to what extent, bell ringing and, say, basket weaving, can be 'mathematical' is one studied by some mathematical ethnographers like Ascher (1991).

relevant for those involved in higher school mathematics. Within this account I have argued for the objectivity of some mathematical results as a function of realist ontology. What underlies the desire to put forward this view is the following moral principle:

☆ *there should (morally) be the possibility for anyone to engage with mathematical ideas*

In other words, the research which ensued from my interpretation of this principle was to develop theoretical ideas about the nature of mathematics which did not demand absolutely that mathematical knowledge was only a cultural artefact. Given the principle stated, I was motivated to make a case for why mathematics is not *just* a function of the practitioners. I hold the view that even though the majority-ethnicity male middle-class bastions of mathematical community privilege are extremely hard to penetrate for someone from outside, this does not imply that those within that community created mathematical knowledge in the sense that they can decree what is to be true. While it is true that they have invented notations, encouraged particular questions - ballistics has been a favourite over the centuries - and chosen new members of their elite group, they did not create the possibility of perceiving discrete entities, from which basic counting starts prior even to language development, nor did they form the human powers used to develop these initial ideas mathematically.

I have argued for the non-relativism of mathematical truths at the higher school mathematics level. This was motivated by the ethical principle discussed. But, even if I may have persuaded the reader that there is something objective about the nature of mathematics, a story about the nature of mathematics is less objective; it is inevitably shaped by the narrator. Truth of some mathematical propositions may be as true as completely as the meaning of 'true' can be taken, but the truth of the proposition 'some mathematical propositions are completely true' is not as firm. I have wanted to express and argue the thesis that there was some aspect of mathematics which was intrinsic to the physicality of human functioning. As another of my former students, Toby Martin, said of mathematics "it seems that maths springs from the world we find ourselves in. Perhaps the way we think is tutored by the world we're in, so the maths we think is obviously going to fit in."

Chapter 8: Conclusion

A contribution made by this thesis is to show the relevance of schools of thought, other than conceptualism, to mathematics in education. I have started from 'mathematical experience', or as Maddy calls it "the phenomenon of practice" (Maddy, 1990, p3), and tried to explain what it is to 'come to experience that practice' (i.e., 'come to know mathematics') through a person-centred philosophical realism. The connection with the practice of teaching mathematics is also important in locating the purpose of this work. And the root purpose of teaching is one which transcends discipline boundaries. I end with the expression of such purpose given by the teacher and writer bell hooks (1994, p207):

“The classroom, with all its limitations, remains a location of possibility. In that field of possibility we have the opportunity to labor for freedom, to demand of ourselves and our comrades, an openness of mind and heart that allows us to face reality even as we collectively imagine ways to move beyond boundaries, to transgress. This is education as the practice of freedom.”



9. Appendices

9.1 For chapter 5

Attached: student work photocopies of their 'angle in semi-circle' assignment

9.2 For chapter 6

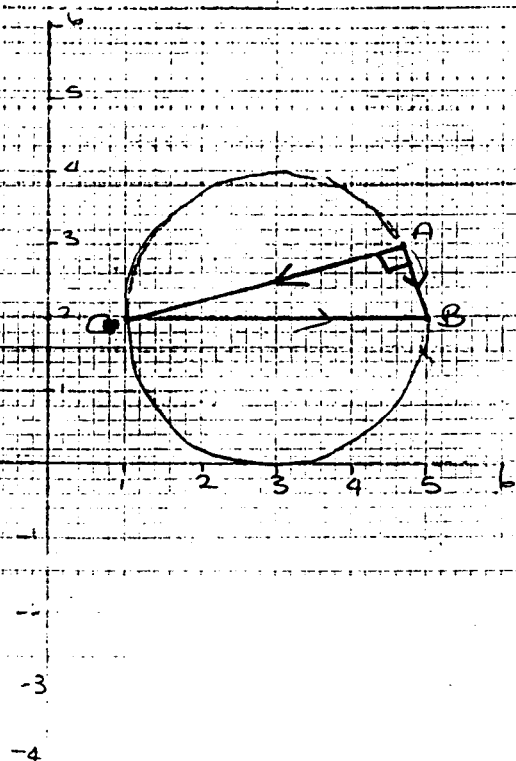
Attached: student work photocopies of their negative numbers -'how I remember' poster and their directed numbers mini-assessment.

9.1

Use of vectors to show angle in semi-circle is a right angle.

7,

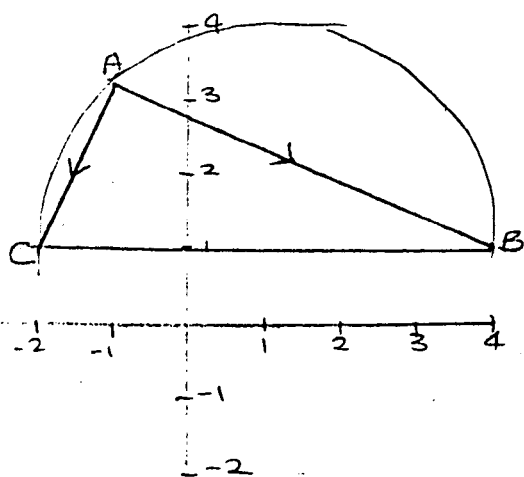
2a(i)



$$\vec{AC} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$$

$$\vec{AB} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

2b(i)



$$\vec{AB} \approx \begin{pmatrix} 5 \\ -2.2 \end{pmatrix}$$

$$\vec{AC} \approx \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

2a. $\vec{AC} = \begin{pmatrix} -3.7 \\ -1 \end{pmatrix}$

$\vec{AB} = \begin{pmatrix} .3 \\ -1 \end{pmatrix}$

Refer to graph paper

$$\vec{AB} \cdot \vec{AC} = \begin{pmatrix} .3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} -3.7 \\ -1 \end{pmatrix} = -1.11 + 1 = -0.11$$

$$-0.11 = |\vec{AB}| |\vec{AC}| \cos \theta$$

$$|\vec{AB}| = \sqrt{.3^2 + -1^2} = \sqrt{1.09} = 1.044$$

$$|\vec{AC}| = \sqrt{-3.7^2 + -1^2} = \sqrt{14.69} = 3.832753579$$

$$\frac{-0.11}{\sqrt{1.09} \sqrt{14.69}} = \cos \theta$$

$$-0.02749 = \cos \theta$$

$$\theta = 91.5$$

$$\theta \approx 90^\circ$$

My calculations will be out slightly^{out} because of the accuracy of my drawings and my guessing the vector to 1dp.

2b) $\vec{AB} \approx \begin{pmatrix} -1 \\ -2 \end{pmatrix}$ $\vec{AC} \approx \begin{pmatrix} 5 \\ -2.2 \end{pmatrix}$

$$\vec{AB} \cdot \vec{AC} = \begin{pmatrix} -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 5 \\ -2.2 \end{pmatrix} = -5 + 4.4 = -0.6$$

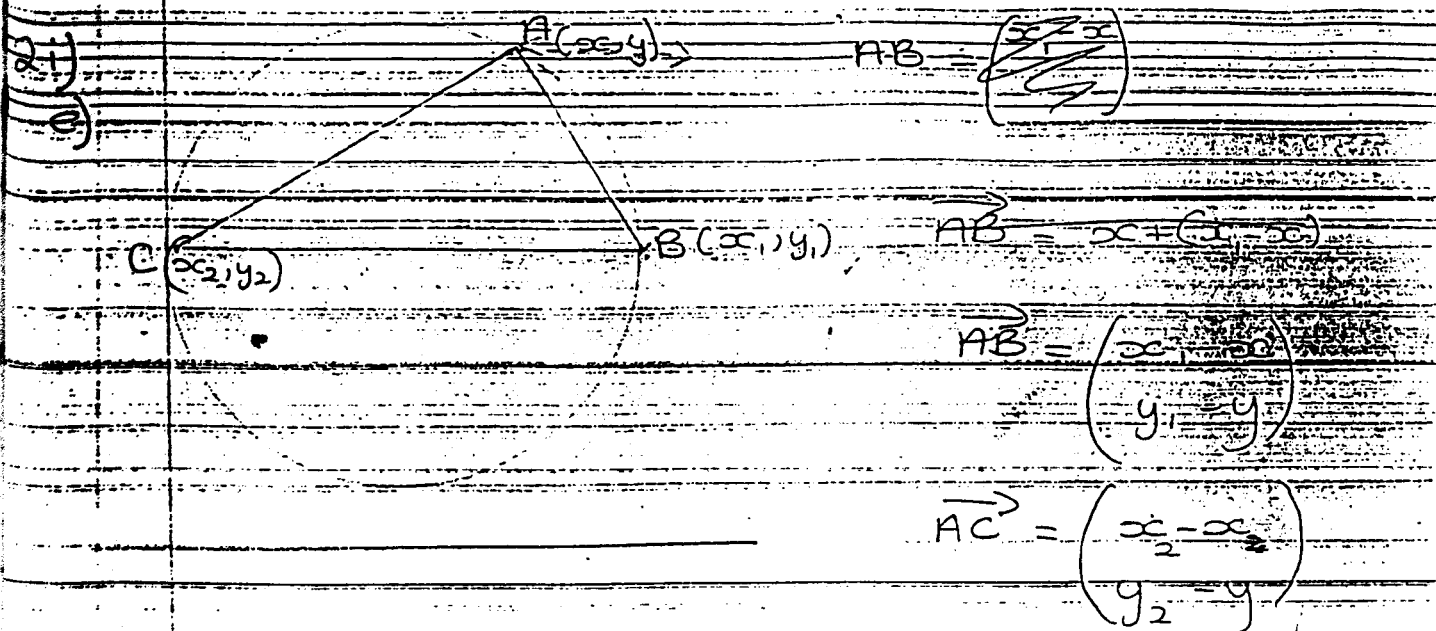
$$-0.6 = |\vec{AB}| |\vec{AC}| \cos \theta$$

$$|\vec{AB}| = \sqrt{-1^2 + -2^2} = \sqrt{5}$$

$$|\vec{AC}| = \sqrt{5^2 + -2.2^2} = \sqrt{29.84}$$

$$\frac{-0.6}{\sqrt{5} \sqrt{29.84}} = \cos \theta \quad \theta \approx 92.8^\circ \quad \theta \approx 90^\circ$$

Attempted proof.



$$\text{subst } z = x_2 - x$$

$$\text{subst } y = y_1 - y$$

$$\vec{AB} \cdot \vec{AC} = \begin{pmatrix} x_1 - x \\ y_1 - y \end{pmatrix} \cdot \begin{pmatrix} x_2 - x \\ y_2 - y \end{pmatrix} = (x_1 - x)(x_2 - x) + (y_1 - y)(y_2 - y)$$

$$\text{subst } a = x_1 - x$$

$$b = y_1 - y$$

$$c = x_2 - x$$

$$d = y_2 - y$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = ac + bd$$

$$(ac + bd) = |\vec{AB}| |\vec{AC}| \cos \theta$$

$$|\vec{AB}| = \sqrt{a^2 + b^2}$$

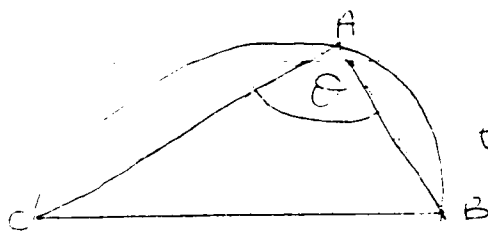
$$|\vec{AC}| = \sqrt{c^2 + d^2}$$

$$\frac{ac + bd}{\sqrt{a^2 + b^2} \sqrt{c^2 + d^2}} = \cos \theta \cdot \frac{ac + bd}{\sqrt{(a+b)^2 - 2ab} \sqrt{(c+d)^2 - 2cd}}$$

Use of Trigonometry to show angle in a semi-circle is a right angle

2ii) With using lined paper I do not expect to get exactly 90° because it will not be accurate enough but anything close to that I can say is approximately 90°

a)



Using a ruler to measure the lines I found that the lengths are approximately

$$AB = 2.75 = a$$

$$BC = 5.42 = c$$

$$AC = 4.7 = b$$

Using the cos rule.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

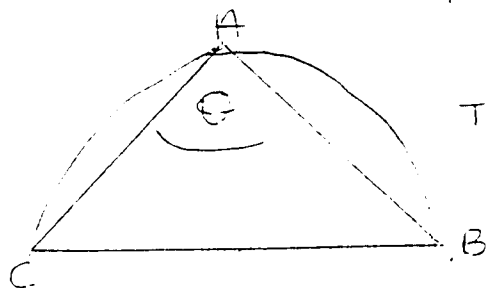
$$29.3769 = 22.69 + 7.5625 - 25.95 \cos A$$

$$-0.2761 = -25.95$$

$$0.0106 = \cos A$$

$$A = 89.4 \quad A \approx 90^\circ$$

b)



The lengths are approximately

$$AB = 4.05 = c$$

$$BC = 5.45 = a$$

$$AC = 3.62 = b$$

$$\frac{a^2 - (b^2 + c^2)}{-2bc} = \cos A$$

$$\frac{0.7431}{-29.322} = \cos A$$

$$-0.025 = \cos A$$

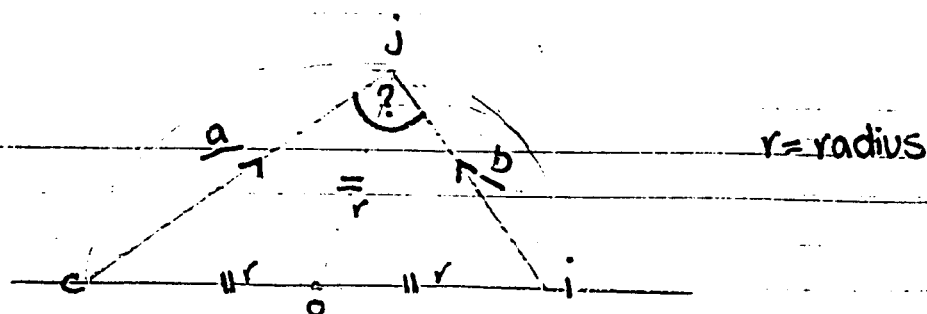
$$A = 91.4 \quad A \approx 90^\circ$$

9.1

E,

Angle in a semicircle

2.i)



As any line from the centre point to the circle is the radius the three lines marked will be the same length. I have named the radius r . To prove that \hat{abc} is a right angle I must prove that \underline{a} is normal to \underline{b} .

I have found \underline{a} by travelling \vec{CO} then \vec{OJ}

\underline{b} has been found by going from \vec{JO} & \vec{OI} .

$$\underline{a} = r\mathbf{i} + r\mathbf{j}$$

$$\underline{b} = -r\mathbf{i} + r\mathbf{j}$$

If \underline{a} is normal to \underline{b} then θ will $= 90^\circ$

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos \theta$$

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos 90$$

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \times 0$$

$$\underline{a} \cdot \underline{b} = 0$$

$$\underline{a} \cdot \underline{b} = (r\mathbf{i} + r\mathbf{j}) \cdot (-r\mathbf{i} + r\mathbf{j})$$

$$= r^2 - r^2$$

$$= 0$$

$\therefore \underline{a}$ is at right angles with \underline{b} .

This will work with any radius as the values of i and j will always be the same, but one will always

9.1

be negative and therefore produce 0.

Example (radius = 3)

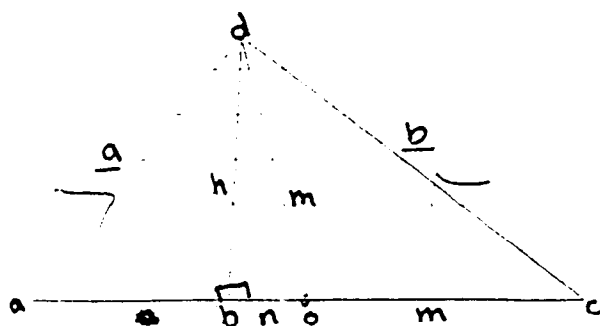
$$(3i + 3j)(-3i + 3j)$$

$$= -9 + 9$$

$$= 0$$

Looking back at my work I realized that the proof I had given would only prove a right angle for 1 triangle in the semicircle, the isosceles triangle. This was because I had use j , which is always vertical, for the line joining the centre and the top of the triangle.

I have worked out an alternative proof ~~which~~ without i and j . I have remained with the $a \cdot b$ proof as I have already proved that it shows when a & b are at right angles with each other. I have just used the height and radius for my proof. This will work for any triangle.



let radius = m
length between ~~base~~ $base$ = n

$$m^2 = h^2 + n^2$$

$$\underline{a} = \begin{pmatrix} m-n \\ h \end{pmatrix}$$

$$\underline{a} = m-n+h \quad \underline{b} = -m-n+h$$

$$\underline{b} = \begin{pmatrix} -(m+n) \\ h \end{pmatrix}$$

$$\underline{a} \cdot \underline{b} = (m-n+h) \cdot (-m-n+h)$$

$$\underline{a} \cdot \underline{b} = -m^2 + n^2 + h^2$$

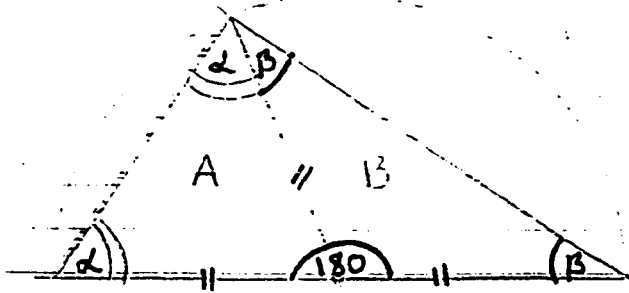
replace $-m^2 \rightarrow h^2 = m^2 - n^2$ (pythagoras) $\rightarrow -m^2 = -h^2 - n^2$

$$\underline{a} \cdot \underline{b} = -h^2 - n^2 + n^2 + h^2$$

$\underline{a} \cdot \underline{b} = 0 \therefore$ angle ~~at~~ $\hat{a}dc$ is a right angle.

any other
(radius)

2ii)



Any line from the centre point to the edge of the circle is the radius and therefore equal to the other lines joining centre and circle. Where the equal lines are part of a triangle the corresponding angles will be equal as well.

Example



Therefore in triangle A the two corners are equal as the adjacent sides are. I have named this angle α .

The same applies in triangle B, which I have named its equal angles β .

All angles in ~~any~~ any triangle add up to 180° :-

$$\alpha + \beta + \alpha + \beta = 180$$

$$\alpha + \beta = \alpha + \beta$$

$$\alpha + \beta = \frac{180}{2}$$


$$\alpha + \beta = 90^\circ = \text{a right angle}$$

This proves that the top corner in the semicircle is a right angle because it contains both $\alpha + \beta$.

Bryony

HOW I REMEMBER DIRECTED NUMBERS!

This is a poster all about how I remember directed number sums.

Here is one way. Imagine there is a mountain like this  If you measure the top height and the water level you can work out whats in between.

Another way to find something out for the multiplication sum is if you get a sum like 2×3 with 2 minuses I always think it can not go any lower so it therefore must go higher $2 \times 3 = 6$

One more way to work out a directed sum is by drawing a number line, like the one below, It really helps to find the number in the gap.

A Number Line.



Bryony

Directed Numbers Mini Assessment

- A) Make up four questions (and put the answer!) that show how well you can add and subtract directed numbers.

1) $-2 - 1 = -3$ oops

2) $1 - 2 = -1$ ✓

3) $3 - 1 = 2$ x 3.

4) $-1 - -1 = 0$ ✓

$1 - 4 = -3$

$1 + 0 = 1$

$-1 - -4 = 3$

$1 - 1 = 0$

- B) Make up four questions (and put the answer!) that show how well you can multiply directed numbers.

1) $2 \times -3 = -6$ ✓

2) $3 = 3$ ✓

3) $-4 \times 5 = -20$ ✓

4) $-1 \times -2 = 2$ ✓

$3 \times 4 = 12$

$5 \times 5 = 25$

$-4 \times 5 = -20$

good!

- C) Now, have a go at these division questions:

Remember to use multiplication to check back.

Example $24 \div 8 = ?$

$24 \div 8 = 3$ because $3 \times 8 = 24$.

✓ 1) $8 \div -2 = ?$ -4 ✓

✓ 2) $-12 \div -4 = ?$ 3 ✓

✓ 3) $-15 \div 3 = ?$ -5 ✓

2 4) $-30 \div (-2 \times -6) = ?$ 3 x 5

$-8 \div -2 = 4$

$-8 \div 4 = -2$

$-12 \div 2 = -6$

$-12 \div 4 = -3$

- D) What do you understand best about directed numbers?

I understand that positive and negative numbers are used to represent different directions or values. For example, in a number line, positive numbers are to the right of zero and negative numbers are to the left of zero.

- E) On what aspect of directed numbers do you feel you need more lessons?

I need more lessons on how to multiply and divide directed numbers, especially when there are multiple operations involved. I also need more practice with word problems involving directed numbers.

HOW I REMEMBER

How To Work Out Directed Numbers

Remember the signs

$$-12 - (-16) = 28$$

$$10 - 15 = -5$$

A positive and a positive equals
a positive
A negative and a positive equals
a negative

A negative and a positive

equals a negative!!!!

Directed Numbers Mini Assessment

A) Make up four questions (and put the answer!) that show how well you can add and subtract directed numbers.

- 1) $26 - 28 = -2$ ✓
- 2) $-12 + -12 = -24$ ✓
- 3) $-55 - -56 = 1$ ✓ good
- 4) $5 + -2 = 3$ ✗ $= -7$

B) Make up four questions (and put the answer!) that show how well you can multiply directed numbers.

- 1) $20 \times -2 = -40$ ✗ $= -40$
- 2) $-3 \times -5 = 15$ ✓
- 3) $-25 \times 2 = -50$ ✓
- 4) $-16 \times -3 = 48$ ✓

C) Now, have a go at these division questions:

Remember to use multiplication to check back.

Example $24 \div 8 = ?$

$24 \div 8 = 3$ because $3 \times 8 = 24$.

- 1) $8 \div -2 = ?$ ✗ ✓ $= -4$ ✓
- 2) $-12 \div -4 = ?$ 3 ✓
- 3) $-15 \div 3 = ?$ 3 ✓ $= -4 \times -5$ ✓
- 4) $-30 \div (-2 \times -6) = ?$ -2.5

D) What do you understand best about directed numbers?

I don't really know because there's little bits of each ones I think that I am good at and not good at.
OK

E) On what aspect of directed numbers do you feel you need more lessons?

The dividing directed numbers is quite hard so I'm not really sure if I got the correct.
I also need to touch up a bit on the \times - + + 's.
On

HOW TO UNDERSTAND + WORK OUT POSITIVE AND NEGATIVE NUMBERS.

NEGATIVE + NEGATIVE = POSITIVE

HOW TO REMEMBER IT.

There are lots of different ways but my way is very simple.

2 rights = 1 right

2 negatives = positive

HOW DO YOU REMEMBER POSITIVE + NEGATIVE NUMBERS AND HOW TO WORK THEM OUT?



Negative + Positive = Positive

How to remember it.

Any number above 9 (negative + positive)

always will equal positive.

Neg. Nine + pos. 10 = 1 (positive)

Negative + Positive = Negative

How to remember it.

I remember Negative + Positive = Negative by saying the first number is negative so the end number must be negative with any number up to 9.

Positive + Negative = Positive / Negative.

There are 2 answers to this one because it depends on the negative number. If it is higher than the positive or negative e.g.

$$6 + (-9) = -3$$

If the negative number is lower than the positive = positive e.g.

$$8 + (-5) = 3$$

POSITIVE + POSITIVE = NEGATIVE

How to remember this, it is similar to negative + negative = positive, but it is positive + positive = negative.

2 rights = 1 wrong

Lizzie

Directed Numbers Mini Assessment

A) Make up four questions (and put the answer!) that show how well you can add and subtract directed numbers.

- 1) $-5 + 9 = 4$ ✓ $-50 - (-25) = -25$
- 2) $-7 + 21 = 14$ ✓ $-30 - 5 = -35$
- 3) $-21 + 30 = 9$ ✓ $-100 - (-30) = -70$ $-65 - (-65) = +30$
- 4) $-40 - (-30) = -10$ ✓ $-30 - (-50) = 20$

B) Make up four questions (and put the answer!) that show how well you can multiply directed numbers.

- 1) $-5 \times 9 = -45$ ✓ $-5 \times 9 = -45$
- 2) $3 \times -3 = -9$ ✓ $-10 \times 9 = -90$ $-5 \times 2 = -10$
- 3) $4 \times 5 = 20$ ✓ $-12 \times 12 = -144$ $-20 \times 7 = -140$
- 4) $-2 \times 3 = -6$ ✓ $-4 \times 7 = -28$ $-25 \times 5 = -125$

C) Now, have a go at these division questions:

Remember to use multiplication to check back.

Example $24 \div 8 = ?$

$24 \div 8 = 3$ because $3 \times 8 = 24$.

- 1) $8 \div -2 = ?$ -4 ✓ $8 \div -2 = -4$
- 2) $-12 \div -4 = ?$ 3 ✓ $-12 \div -4 = 3$
- 3) $-15 \div 3 = ?$ -5 ✓ $-15 \div 3 = -5$
- 4) $-30 \div (-2 \times -6) = ?$ 5 ✓ $-30 \div 12 = -2.5$

D) What do you understand best about directed numbers? I understand best that 2 negative numbers = 1 positive number.
 \uparrow depends what you do with them!

E) On what aspect of directed numbers do you feel you need more lessons? I think that more lessons on multiplying directed numbers would be good. OK. And division would be good.

Quenie

THIS IS THE
WAY I
REMEMBER
IT!

is is one of the ways I remember it.

this is another way I remember it.

057 —

30 —

20 —

10 —

0 — 0

-10 —

-20 —

-30 —

400 —

- 60 —

way I remember it with
mountain is is I am climbing
and look down which is
that means I would
going the wrong way so I
fall and kill my self so I
don't want to look back down.

The way I remember the other one
is is I am walking along
a road going north which is
positive and is I stopped and looked
back which is south and started
walking that way I would be going
the wrong way.

Directed Numbers Mini Assessment

A) Make up four questions (and put the answer!) that show how well you can add and subtract directed numbers.

- 1) $300 + 400 = 700$ ✓ $-200 + -300 = -500$
- 2) $100 + 200 = 300$ ✓ $200 + 500 = 700$
- 3) $200 + 300 = 500$ ✓
- 4) $-100 - 200 = -300$ ✓ dare to go below zero!
 $500 - 300 = 200$

B) Make up four questions (and put the answer!) that show how well you can multiply directed numbers.

- 1) $5 \times 2 = 10$ ✓ $2 \times 2 = 4$
- 2) $3 \times 4 = 12$ } hmmm
- 3) $2 \times 3 = 6$ ✓
- 4) $1 \times 1 = 1$ ✓

C) Now, have a go at these division questions:

Remember to use multiplication to check back.

Example $24 \div 8 = ?$

$24 \div 8 = 3$ because $3 \times 8 = 24$.

- 1) $8 \div -2 = ?$ ✓
- ✓ 2) $-12 \div -4 = ?$ ✓
- 3) $-15 \div 3 = ?$ ✓
- 4) $-30 \div (-2 \times -6) = ?$ ✓

D) What do you understand best about directed numbers?

I understand that positive and negative numbers are not an uncommon feeling!

E) On what aspect of directed numbers do you feel you need more lessons?

I need more lessons on how to add and subtract directed numbers. good! keep trying.

Colin

Directed Numbers Mini Assessment

A) Make up four questions (and put the answer!) that show how well you can add and subtract directed numbers.

- 1) $(4 - -2) - 4 = 2$ ✓
- 2) $(-6 + 4) + 4 - -2 = 4$ ✓
- 3) $(5 - 2) - 3 - 4 - -3 = \cancel{4}^4$ check
- 4) $(2 - 5) - 3 - -1 = \cancel{-6}^{-6}$ close

B) Make up four questions (and put the answer!) that show how well you can multiply directed numbers.

- 1) $2 \times -5 = -10$ ✓
- 2) $-6 \times -2 = 12$ ✓
- 3) $7 \times -20 = -140$ ✓
- 4) $-7 \times 2 = -14$ ✓ good

C) Now, have a go at these division questions:

Remember to use multiplication to check back.

Example $24 \div 8 = ?$

$24 \div 8 = 3$ because $3 \times 8 = 24$.

- 1) $8 \div -2 = ?$ -4
- 2) $-12 \div -4 = ?$ 3
- 3) $-15 \div 3 = ?$ -5
- 4) $-30 \div (-2 \times -6) = ?$ -2.5

> Don't understand this

D) What do you understand best about directed numbers?


E) On what aspect of directed numbers do you feel you need more lessons? for things like $6 - -2 = etc$

$6 \div$
?
!

Colin

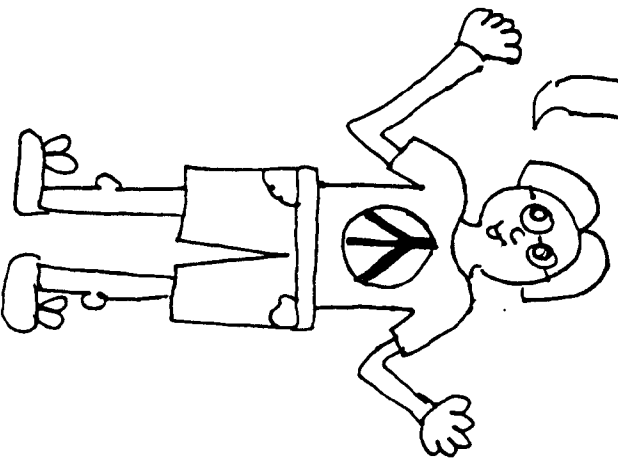
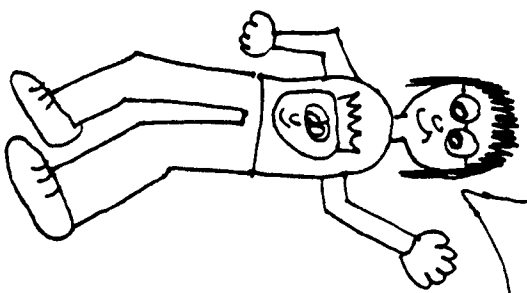
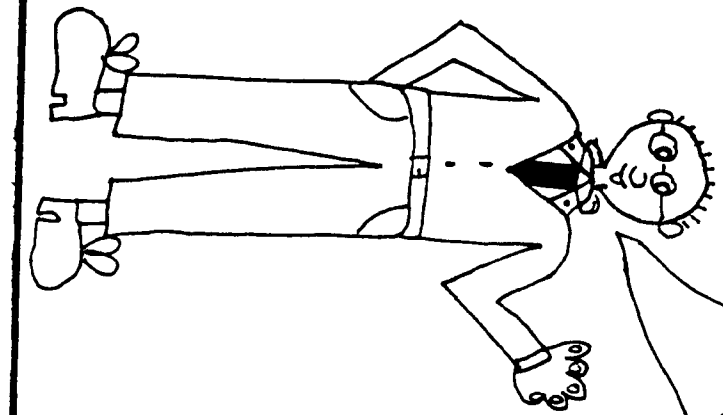
A WAY OF REMEMBERING NEGATIVE NUMBERS

Two negative numbers
with brackets = +

You'll always see
this  in front of
it

Negative numbers
are less than
zero

BERTIE
Digion



Directed Numbers Mini Assessment

A) Make up four questions (and put the answer!) that show how well you can add and subtract directed numbers.

1) $4 - 95 = -91$ ✓

2) $6 - 57 = -51$ ✓

3) $-3 + 5 = -8$ ✗ 2

4) $-3 + 3 = 6$ ✗

$-3 + 3 = 0$

$3 - 61 = -61$

$-2 + 5 = -7$

B) Make up four questions (and put the answer!) that show how well you can multiply directed numbers.

1) $-5 \times 5 = -25$ ✓

2) $-4 \times -4 = 16$ ✓

3) $6 \times -3 = -18$ ✓

4) $-6 \times 5 = 30$ ✓

$-2 \times 3 = -12$

$-5 \times 4 = -20$

$-2 \times 2 = -4$

$-3 \times 5 = -15$

C) Now, have a go at these division questions:

Remember to use multiplication to check back.

Example $24 \div 8 = ?$

$24 \div 8 = 3$ because $3 \times 8 = 24$.

1) $8 \div -2 = ?$ -4 ✓

2) $-12 \div -4 = ?$ 3 ✓

3) $-15 \div 3 = ?$ -5 ✓

4) $-30 \div (-2 \times -6) = ?$ -26 ✓

D) What do you understand best about directed numbers?

That when there is two minus numbers when you add them together you get a positive number. ✓

E) On what aspect of directed numbers do you feel you need more lessons?

multiplication.

OK - (B) is good enough.

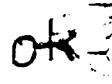
Riding forward at 5 mph in 2 hours time - 10 miles along.

Riding backwards at 5 mph in 2 hours time - 10 miles back

5 x 2 = 10



10 miles along



10 miles back

I think the easiest way to remember this is by drawing a road and going backwards and forwards along the road.

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